Stackelberg or Cournot? A general model of duopolistic competition with endogenous timing.

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Abstract

We study a general duopoly game to show that whether the competition is simultaneous or sequential depends essentially on the *endogenizing game* (the framework to endogenously obtain the timing of movements) and the sub (or super) modularity of the payoff functions. This result challenges the idea that the timing depends on an *intrinsic difference* of the players (such as marginal cost or capacity of production). In particular, we show that when competition is supermodular, the interaction is sequential; and when it is submodular, it can be simultaneous or sequential depending on the considered endogenizing game. Our results allow us to provide a novel answer to the Stackelberg versus Cournot question based on the risk that players face when trying to achieve their desired positions in the market.

JEL codes: L13, C72, D21.

Keywords: Endogenous Timing; Cournot; Stackelberg; Duopolies; Risk Dominance.

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1 Introduction

An essential feature of models of duopolistic competition is whether firms take actions simultaneously or sequentially. In the former case, each firm decides what to do without knowing what the other does, and in the latter, a firm acts as a leader and the other as a follower, implying that the follower knows what the leader did before deciding her own action; and, in turn, the leader takes into account that the follower will have such information before deciding what to do. When firms compete in quantities, these models are commonly known as Cournot and Stackelberg competition respectively.[1] The distinction was first pointed by von Stackelberg (1934) as a critique of the Cournot (1838) equilibrium idea. However, the Stackelberg solution concept for the standard duopolistic competition implies the arbitrary assignment of the roles of leader and follower for players that are, *a priori*, interchangeable. Such discrepancy gave rise to the well-known question of "Stackelberg versus Cournot."

This problem of whether the competition should be modeled as simultaneous or sequential can be addressed from various points of view. Perhaps the most direct way is by describing the preferences (about being leader or follower) of the agents based on a certain characteristic of the model. For instance, Dowrick (1986) studies the preferred role as a function of the slope of the reaction curves, concluding that the players will disagree with the preferred role if their reaction functions are decreasing, and will agree if they are increasing. Okuguchi (1999) derives conditions that make it more advantageous to be a leader or a follower, and shows that if the reaction functions are increasing, the profits of the players in the Cournot competition are lower than those of the Stackelberg setting (for both leader and follower). Also in line with the profit analysis, Amir and Grilo (1999) and Julien (2011) show conditions that allow the followers to achieve higher profits than the leaders. In the case of price competition of firms with different capacities, Furth and Kovenock (1993) and Deneckere and Kovenock (1992) show that the firm with the lowest capacity strictly prefers being a follower.

Going one step further in the analysis, it is natural to think not only about describing preferences but also to model how such preferences translate into a market interaction where players assume the roles of leader or follower. This stream, which is the most relevant approach for the work that we present in this article, is called "the endogenous timing in duopolies literature". The intuition of these models is that, in some cases having an *intrinsic difference* such as lower marginal cost or more capacity of production or greater investment

[1] Although the original Stackelberg model only specifies that competition is sequential, not the competition variable.

Intrinsic difference/Competition variable	Price competition	Quantity competition
Quality	Li (2014) Lambertini & Tampieri (2017)	Lambertini & Tampieri (2011) Jinji (2004)
Location	Lambertini (1997)	
Capacity		Lu & Poddar (2009)
Marginal Cost	Amir et al. (1999) van Damme & Hurkens (2004) Amir & Stepanova (2006)	Amir & Grilo (1999) van Damme & Hurkens (1999)
R&D		Amir et al. (2000) Tesoriere (2008)

Table 1: Endogenous Timing Summary (Price/Quantity)

in R&D than your rival, could give you the *best* role in the market. For models of endogenous timing with price competition, see Lambertini (1997), Amir et al. (1999), van Damme and Hurkens (2004), Amir and Stepanova (2006), Li (2014) or Lambertini and Tampieri (2017). In the case endogenous timing with quantity competition, we have van Damme and Hurkens (1999), Amir and Grilo (1999), Amir et al. (2000), Tesoriere (2008) and Lu and Poddar (2009).

The common strategy of these articles is as follows.

- Consider a so-called "basic interaction", which is the scenario for which they want to endogenously obtain the timing. These basic interactions are usually price or quantity competition plus an intrinsic difference (marginal costs, capacities, process R&D, etc).
- Extend such basic interaction using the Game with Observable Delay (GOD) or the Game with Action Commitment (GAC) from Hamilton and Slutsky (1990). Each equilibrium of the extended game naturally induces a timing of movements in the basic interaction.
- In case of multiple equilibria, refine.[2].

Summarizing, this stream of the literature can be classified as in Table 1, and the most important element for our work is that these articles focus the endogenous timing analysis on the feature that makes firms different (quality, location, capacity, etc), that is, their intrinsic difference.

[2]Typically with the risk dominance concept of Harsanyi et al. (1988), as payoff dominance turns out to be helpless in many of the considered cases.

Contesting that idea, in this paper we show that the timing outcome depends crucially on the model to *endogenize* the movements, what we refer to as the "*endogenizing game*"; and on the sub (or super) modularity of the payoff functions. In other words, if we are trying to determine if competition will be simultaneous or sequential, instead of looking at the intrinsic difference among the players, we should look at the endogenizing game and the slope of the best response functions. Our main contribution is that we provide a novel interpretation of the "Stackelberg versus Cournot" problem based on the market structure, shape of the reaction functions and risk considerations. We also develop a particular case (differentiated quantity competition with different marginal costs) to analyze the differences between GOD and GAC when applied to the same particular basic interaction, something that, to the extent of our knowledge, has not been exhaustively performed in the literature. Our results reinforce the idea that the influence of the best response functions on the endogenous timing outcomes its only through their qualitative behavior (being increasing or decreasing) and not their quantitative characteristics (the "magnitude" of their slopes).

Specifically, we consider a very general duopolistic setting as the basic interaction of the game. The payoff functions can be either submodular or supermodular, which means that competition can be in strategic substitutes or strategic complements. We extend this basic interaction using the two models of Hamilton and Slutsky (1990)[3] and refine the multiple equilibria with the risk dominance concept of Harsanyi et al. (1988). When competition is submodular, we show that firms would like to avoid being a follower, because that is the worst equilibrium scenario in terms of payoff. Our model exploits that idea in the following sense: depending on the market structure (specifically, the "endogenizing game"), trying to avoid the follower position might entail more or less risk for the firms. In particular, given the market conditions, sometimes trying to avoid the follower position implies committing to risky actions and, if both firms do so, they could end up even worse than in the follower position. To avoid this risk, firms might prefer to wait and see what the other does, giving rise to a Stackelberg result (sequential competition). In other cases, the market structure could allow the possibility of avoiding the follower position just by making a statement or declaring an intention of taking an action. In such cases, given that there is no risk involved, firms could end up playing at the same time, that is, a Cournot result (simultaneous competition). For the supermodular case, we show that competition is sequential for both endogenizing games under risk considerations. The intuition behind is that, for supermodular competition, there is a predominant incentive to *wait and see* what the other player does before choosing an action. This behavior could be also understood as a way to avoid the

[3] The Game with Observable Delay (GOD) and the Game with Action Commitment (GAC).

risk of committing to an action and being outperformed by the other player afterwards.

We finally note that another fruitful approach to the "Stackelberg versus Cournot" question has been the efficiency analysis. In this sense, the classical *static* oligopoly literature agrees that the Stackelberg equilibrium is more efficient than the Cournot one (see Amir and Grilo (1999), Robson (1990b), Anderson and Engers (1992) or Huck et al. (2001)). However, in other contexts, this result might not hold. For instance, Haan and Maks (1996) use an entry-deterrence model to show that the prices are not necessarily lower when the post-entry competition is Stackelberg instead of Cournot. Another example is a recent work of Colombo and Labrecciosa (2019) which shows that, from a differential game point of view, the Cournot equilibria could be more efficient (in terms of total surplus) than the Stackelberg one. Although this is an interesting way to analyze the problem, we shall not dwell there.

The following sections of the paper are organized as follows: in section 2, we give a more detailed description of the endogenous timing models and the refinement criterion that we use. In section 3 we present the model, and its extensions using GOD and GAC. In section 4, we discuss our main results and their interpretation. In section 5, we study the particular case of a model of differentiated quantity competition with different marginal costs. Finally, in section 6, we provide the conclusions of the article.

2 Preliminaries

Here we describe the endogenous timing extension models from Hamilton and Slutsky (1990) and the refinement concept from Harsanyi et al. (1988), as their understanding is essential for the developments presented further in the paper.

2.1 Endogenous Timing Models

Frequently, the timing of a game is understood as exogenously given, meaning that the interaction is already defined as simultaneous or sequential, and if it is sequential, the roles of leader and follower are also pre-defined. For instance, in security games it is assumed that the defendant commits to a strategy first, and then the attacker plays in response to that strategy; or in the case of entry-deterrence models, the incumbent is the leader and the entrant is the follower. Nevertheless, since the late '80s, game theorists started to think that the timing of an interaction should not be exogenous but the result of the players' decisions. Two of the most important frameworks to work on *endogenizing* the timing were provided by Hamilton and Slutsky (1990): the Game with Observable Delay and the Game with Action Commitment (GOD and GAC from now on). The general idea of both models is the following: for a basic interaction (namely, price or quantity competition), they define an extended game by adding a pre-play stage, in which the players have to choose an action related to the timing; this action is indeed taken simultaneously. Then, when the equilibria of the extended game are computed, each one of those equilibria naturally induces a timing of movements on the basic interaction. This work of course is not the only one aiming at the same target. There are continuum time approaches in Robson (1990a), Deneckere and Kovenock (1992), Furth and Kovenock (1993) and Meza and Tombak (2009). However, we focus on the endogenous timing models of Hamilton and Slutsky (1990) because of their insightful approach and significant influence on posterior articles.

In the following two subsections, we describe the action space on the pre-play stage for both GOD and GAC, and also detail how the game is played after those decisions. We might refer to GOD and GAC as the *endogenizing games* from now on.

2.1.1 Game with Observable Delay

In the pre-play stage of the Game with Observable Delay players can choose an action from the set $\{E, L\}$, where E stands for *early* and L stands for *late*. If both players choose the same action, they play a simultaneous game in the basic interaction. If one of them chooses E and the other one chooses L, they play a sequential game in which the former player is the leader and the latter is the follower. Assuming that players are rational, the payoffs in the basic interaction are completely determined by the decisions made in the pre-play stage as follows:

- If both players choose the same action, they obtain their Nash Equilibrium payoffs in the basic interaction.
- If a player chooses E and the other chooses L, the former obtains her leader equilibrium payoff and the latter her follower equilibrium[4] payoff.

2.1.2 Game with Action Commitment

In the pre-play stage of the Game with Action Commitment, the action space is not necessarily finite as in the GOD model. In this case, each player can either commit to an action

[4]Subgame Perfect Equilibrium.

or *wait*, that is, the action space in the pre-play stage for each player is $A \cup \{W\}$, where A is the action space in the basic interaction and W represents the option of *waiting*. According to the decisions made in this pre-play stage, the basic interaction is as follows:

- If both players commit to an action, they play simultaneously in the basic interaction and must use those chosen actions (that is the commitment part).
- If one player commits to an action and the other waits, they play sequentially in the basic interaction with the former being leader (playing that action) and the latter being follower (playing her best response after learning the action of the leader).
- If both players wait, they play simultaneously in the basic interaction.

In each case, players obtain the corresponding payoffs.

2.2 Refinement Concept

When applying GOD or GAC to basic interactions, it is frequent to find multiple equilibria of the extended games, and therefore a multiplicity of possible endogenous timing outcomes. Since we are trying to determine which type of competition, sequential or simultaneous, will arise, the use of a refinement method becomes necessary. Perhaps the first idea that comes to mind is to use payoff dominance. Unfortunately, because of the nature of the interactions, in many cases it is not possible to establish that relationship. An alternative approach is the risk dominance idea from Harsanyi et al. (1988), which has very interesting properties and allows to provide insightful interpretations. The idea is that, since players do not know which (of possible many) equilibria is going to be played, they will measure the risk involved in playing each one and they will coordinate expectations on the less risky one, which is then called the risk dominant equilibrium.

For 2×2 games, the risk dominant equilibrium is the one that has the biggest deviation losses. To fix ideas, let us consider example in Table 2.

	A	B
A	80,80	80,0
B	0,80	100,100

Table 2: Risk versus payoff dominance.

This game has two equilibria in pure strategies: (A, A) and (B, B). Note first that (B, B) is the payoff dominant equilibrium. On the other hand, if we denote by L_s the deviation losses associated to equilibrium s, we have that:

$$L_{(A,A)} = (80 - 0) \times (80 - 0) \qquad \text{and} \qquad L_{(B,B)} = (100 - 80) \times (100 - 80).$$

Therefore, the risk dominant equilibrium is (A, A).

For richer games, the definition of risk dominance is much more complicated and requires two previous concepts: the *bicentric prior* and the *tracing procedure*. The bicentric prior describes the initial assessment of the players about the situation. If this initial belief is not an equilibrium, it can not be the final perception about how the game should be played. In such a case, the players must adjust their plans until they are in equilibrium. The tracing procedure models this adjustment process.

Formally, let $g = (S_1, S_2, u_1, u_2)$ be a game of two players with strategy spaces S_1 and S_2, and payoff functions u_1 and u_2. Suppose that there are two equilibria in g: $s = (s_1, s_2)$ and $s' = (s'_1, s'_2)$. The initial beliefs are constructed as follows:

- Player 2 believes that 1 will play s_1 with probability z_2 and s'_1 with probability $(1 - z_2)$. Therefore, 2 will play a best response against that belief. Denote this best response by $b_2(z_2)$.

- Player 1 does not know z_2, then she assumes that z_2 is uniformly distributed. Consequently, if we consider a random variable $Z_2 \sim U([0, 1])$, we can state that 1 will believe she is playing against $m_2 = b(Z_2)$. This m_2 is the *prior belief* of player 1 about 2's behaviour.

- Analogously, we can construct the prior belief m_1 of player 2 about 1's behavior.

- The pair $m = (m_1, m_2)$ is called the *bicentric prior* associated to s and s'.

Regarding the tracing procedure, it is a map converting initial beliefs into equilibria of the game. Let m_i be a mixed strategy for player i. This mixed strategy represents the initial uncertainty of player j about i's behavior at the beginning of the game. For a pair of mixed strategies $m = (m_1, m_2)$, and every $t \in [0, 1]$, it is possible to define a perturbed game $g^{t,m} = (S_1, S_2, u_1^{t,m}, u_2^{t,m})$ which has the same strategies and players of g, but different payoff functions. Specifically, these new payoffs are given by:

$$u_i^{t,m}(s_i, s_j) = (1-t)u_i(s_i, m_j) + tu_i(s_i, s_j),\ i = 1, 2. \tag{1}$$

Observation 1 *For $t = 0$, the payoff of each player depends only on her own behaviour and prior belief. For $t = 1$, the game $g^{t,m}$ coincides with g.*

For this family of perturbed games, it is possible to define the equilibrium correspondence graph Γ^m, which is the representation of the equilibria of the perturbed game for each value t:

$$\Gamma^m \doteq \{(t, s) :\ t \in [0, 1],\ s \text{ is an equilibrium of } g^{t,m}\}.$$

Suppose again that the original game has two equilibria: s and s'. If players have initial beliefs about their opponent's actions given by m, and they adjust these beliefs according to the tracing procedure in terms of $u^{t,m}$, eventually converging to s, it will be said that *s risk dominates s'*. If the procedure converges to s', we have the opposite situation. If the process does not reach either s or s', it is not possible to establish a comparison in terms of risk.

Definition 1 *If $s^{1.m} = s$, the equilibrium s risk dominates s'. Analogous if $s^{1.m} = s'$. If $s^{1,m} \neq s, s'$, neither of the equilibriums risk dominates the other one.*

Observation 2 *This relation between equilibria is not necessarily transitive. This is, if s risk dominates s', and s' risk dominates s'', it could happen that s would not risk dominate s''.*

3 The model

Consider two firms, namely $i = 1, 2$, that compete in variables $x_1 \geq 0$ and $x_2 \geq 0$.[5] After players choose their actions, they obtain payoffs

$$\Pi_i(x_1, x_2), \quad i = 1, 2.$$

We assume that these payoff functions are $\mathcal{C}^2(\mathbb{R} \times \mathbb{R})$ and concave in the own variable. Our main assumptions throughout the article will be related to their supermodularity or submodularity.

[5]Note that we make no assumption about what these variables are.

Definition 1 *We say that competition is supermodular if the payoffs are supermodular on competition variables. This is, if for all* $x_1, x_2 \geq 0$*:*

$$\text{(A1)} \qquad \frac{\partial^2 \Pi_i}{\partial x_i \partial x_j}(x_1, x_2) \geq 0, \;\; i, j = 1, 2; \;\; i \neq j. \tag{2}$$

Definition 2 *We say that competition is submodular if the payoffs are submodular on competition variables. This is, if for all* $x_1, x_2 \geq 0$*:*

$$\text{(A2)} \qquad \frac{\partial^2 \Pi_i}{\partial x_i \partial x_j}(x_1, x_2) \leq 0, \;\; i, j = 1, 2; \;\; i \neq j. \tag{3}$$

These notions of supermodular and submodular competition are a generalization of the well-known concept of strategic complements and strategic substitutes. In particular, when competition is supermodular, best responses are increasing; and when is submodular, best responses are decreasing.

Consider the three possible subgames in this basic interaction: simultaneous play, sequential with player 1 as the leader, and sequential with player 2 as the leader. We assume that each one of these subgames has a unique equilibrium and that those equilibria are different from each other. We also use the following notation:

- In the case that player 1 is the leader, x_1^L and x_2^F are the equilibrium actions for player 1 and 2 respectively.
- When the leader is player 2, x_1^F and x_2^L are the actions.
- Finally, in the case of the simultaneous moves, the equilibrium actions are denoted by x_1^N and x_2^N.

The upper indexes F, L, and N represent the follower, leader, and Nash (simultaneous) equilibrium actions respectively.

In the next two subsections, we extend this basic interaction using the GOD and GAC models from Hamilton and Slutsky (1990), and refine the possible multiple equilibria with the risk dominance concept from Harsanyi et al. (1988).

3.1 Extension using GOD

Recall that players have two possible actions in the pre-play stage: to move early (E) or to move late (L). Consequently, the normal form of the extended game can be represented as in Table 3.

	E	L
E	Π_1^N, Π_2^N	Π_1^L, Π_2^F
L	Π_1^F, Π_2^L	Π_1^N, Π_2^N

Table 3: GOD Extension model.

Where $\Pi_1^N \doteq \Pi_1(x_1^N, x_2^N)$, $\Pi_1^L \doteq \Pi_1(x_1^L, x_2^F)$ and $\Pi_1^F \doteq \Pi_1(x_1^F, x_2^L)$. Analogous for player 2.

Definition 3 *We say that a player has a first mover advantage if her leader equilibrium payoff is greater than her follower equilibrium payoff. On the other hand, we say that a player has a second mover advantage if her follower equilibrium payoff is greater than her leader equilibrium payoff.*

Before presenting the main results of this subsection, we prove Lemma 1, which provides sufficient conditions regarding the preferred role of the players.

Lemma 1 *Let us assume that payoffs are monotone as a function of the action of the other player.*

1 If $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} > 0,\ i, j = 1, 2,\ i \neq j,$ *and*

1.1 $x_j^N > x_j^L$, *player* i *has a first mover advantage.*

1.2 $x_j^L > x_j^F$, *player* i *has a second mover advantage.*

2 If $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} < 0\ i, j = 1, 2,\ i \neq j$ *and*

2.1 $x_j^N < x_j^L$, *player* i *has a first mover advantage.*

2.2 $x_j^L < x_j^F$, *player* i *has a second mover advantage.*

The proof of Lemma 1 can be found in the Appendix. From such proof, we also obtain that, when a player has a first mover advantage, her simultaneous equilibrium payoff is strictly greater than her follower equilibrium payoff.

Now we present the endogenous timing results for this extension model. Theorem 1 says that, if competition is supermodular and the payoffs are both monotone increasing or decreasing as a function of the action of the other player, at least one of them has a second mover advantage, This implies sequential play in the basic interaction.

Theorem 1 *Considering (A1) as in 2, and assuming that* $\dfrac{\partial \Pi_1(x_1, x_2)}{\partial x_2}$ *and* $\dfrac{\partial \Pi_2(x_1, x_2)}{\partial x_1}$ *are both positive or negative for all* $x_1, x_2 \geq 0$*, there are two possible endogenous timing equilibrium outcomes of the extended game:* (E, L) *and* (L, E)*. Both imply sequential play in the basic interaction.*

Proof: We use Lemma 1 to show that at least one player has a second mover advantage. For player 1, we have:

$$\begin{aligned} \Pi_1(x_1^L, x_2^F) &> \Pi_1(x_1^N, x_2^N) \\ &\geq \Pi_1(x_1^L, x_2^N). \end{aligned}$$

Analogously, we can prove the same for player 2. Summarizing, we have:

$$\Pi_1(x_1^L, x_2^F) > \Pi_1(x_1^L, x_2^N), \tag{4}$$

and

$$\Pi_2(x_1^F, x_2^L) > \Pi_2(x_1^N, x_2^L). \tag{5}$$

As we are assuming that payoffs are monotone as a function of the rival's action, we can divide our analysis in two cases.

- If $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} > 0, \; i, j = 1, 2, \; i \neq j$, then from (4):

$$x_2^F > x_2^N.$$

Which, in turn, implies that:

$$x_1^L > x_1^N,$$

because the best responses are increasing. Using (5), we can obtain the analogous result for the actions of player 2. Thus, we have that both the leader and follower equilibrium actions are above the simultaneous equilibrium action for both players. Considering this fact, and that the best response functions are increasing, there are three possible cases and we analyze each one using Lemma 1:

(i) If $x_1^L > x_1^F$ and $x_2^L > x_2^F$, both players have a second mover advantage.

(ii) If $x_1^L < x_1^F$ and $x_2^L > x_2^F$, player 1 has a second mover advantage.

(iii) If $x_1^L > x_1^F$ and $x_2^L < x_2^F$, firm 2 has a second mover advantage.

- Let us assume now that $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} < 0 \;\; i, j = 1, 2, \;\; i \neq j$. Again, from (4), we have that:

$$\Pi_1(x_1^L, x_2^F) > \Pi_1(x_1^L, x_2^N) \Rightarrow x_2^F < x_2^N \Rightarrow x_1^L < x_1^N.$$

Using (5), we can prove the same for player 2. Therefore, both the leader equilibrium action and the follower equilibrium action are below the simultaneous equilibrium action. Again, this lead us to study three cases, based on Lemma 1.

(i) If $x_1^L < x_1^F$ and $x_2^L < x_2^F$, both players have a second mover advantage.

(ii) If $x_1^L > x_1^F$ and $x_2^L < x_2^F$, player 1 has a second mover advantage.

(iii) If $x_1^L < x_1^F$ and $x_2^L > x_2^F$, player 2 has a second mover advantage.

■

Now, we study the submodular case. In particular, Theorem 2 says that the endogenous timing outcome is simultaneous competition, provided the same monotonicity condition of Theorem 1.

Theorem 2 *Considering assumption (A2) as in 3, and assuming that* $\dfrac{\partial \Pi_1(x_1, x_2)}{\partial x_2}$ *and* $\dfrac{\partial \Pi_2(x_1, x_2)}{\partial x_1}$ *are both positive or negative for all* $x_1, x_2 \geq 0$ *there is a unique endogenous timing equilibrium outcome of the extended game:* (E, E)*. This is, simultaneous play in the basic interaction.*

Proof: Recall first that inequalities (4) and (5) from the proof of Theorem 1 are still valid in this case, since they are consequence only of the definition of best response.

- Let us assume that $\frac{\partial \Pi_i(x_i, x_j)}{\partial x_j} < 0, \; i, j = 1, 2, \; i \neq j$. From inequality (4), we have that:

$$x_2^N > x_2^F.$$

Which, since best response functions are decreasing, implies that:

$$x_1^N < x_1^L.$$

Analogously, using (5):

$$x_1^N > x_1^F \text{ and } x_2^N < x_2^L.$$

Summarizing, we have the following sequence of inequalities for players 1 and 2:

$$x_1^F < x_1^N < x_1^L \text{ and } x_2^F < x_2^N < x_2^L,$$

Considering Lemma 1, the result follows.

- Assume now that $\frac{\partial \Pi_i(x_i, x_j)}{\partial x_j} > 0, \; i, j = 1, 2, \; i \neq j$. Analogously as we did in the previous case, we obtain that:

$$x_1^F > x_1^N > x_1^L \text{ and } x_2^F > x_2^N > x_2^L,$$

which proves the result .

■

A noteworthy element of Theorem 1 and Theorem 2 is that they do not depend on firms having (or not having) an intrinsic characteristic that could make them different from each other. In particular, the fact that competition in the basic interaction results in simultaneous or sequential play only depends on the slope of the best response functions.

Recall that our main interest in this article is to understand the endogenous timing outcomes for models that are *symmetrical*, in the sense that the supermodular or submodular condition is valid for both players, and also that both payoffs are monotone increasing or decreasing as a function of the action of the other player. In other words, cases in which one player understands the interaction as submodular and the other as supermodular, or when the payoff of one player is increasing as a function of the rival's action and that of the other player is decreasing, are not our primary interest. Nevertheless, covering such cases can provide more intuition, and therefore we do that analysis in the remainder of this section.

In Theorem 3, we assume that the payoffs are not simultaneously increasing or decreasing as a function of the rival's action. In that scenario, if competition is supermodular, the endogenous timing outcome is simultaneous play; and if it is submodular, the result is sequential play.

Theorem 3 *Assume, without loss of generality, that* $\dfrac{\partial \Pi_1(x_1, x_2)}{\partial x_2} > 0$ *and* $\dfrac{\partial \Pi_2(x_1, x_2)}{\partial x_1} < 0$ *for all* $x_1, x_2 \geq 0$.

- *If (A1) on 2 holds, the endogenous timing equilibrium outcome is* (E, E).
- *If (A2) on 3 holds, the endogenous timing equilibrium outcomes are* (E, L) *and* (L, E).

The proof can be found in the Appendix.

The last result of this section is Theorem 4, which covers the case in which one player sees the interaction as supermodular, and the other one as supermodular. We find that the endogenous timing outcome is sequential, and even more, the player that has submodular payoff becomes the leader. The proof can be also found in the Appendix.

Theorem 4 *Let us assume that* $\dfrac{\partial \Pi_1(x_1, x_2)}{\partial x_2}$ *and* $\dfrac{\partial \Pi_2(x_1, x_2)}{\partial x_2}$ *are both positive or negative for all* $x_1, x_2 \geq 0$. *Assume also that (A1) is valid for player 1 and (A2) for player 2. Then, the endogenous timing outcome is* (L, E).

We postpone the discussion of these results until section 4. Now, we move on to the GAC extension model.

3.2 Extension using GAC

First, we present an existence result saying that the endogenous timing outcome can be either simultaneous or sequential. Then, we refine these multiple equilibria using the risk dominance concept from Harsanyi et al. (1988), considering separated analysis for the supermodular and submodular case.

Theorem 5 *Under either assumption (A1) or (A2), as in 2 and 3, there are three equilibrium profiles supported by SPE of the extended game:* (x_1^L, W_2), (x_1^N, x_2^N) *and* (W_1, x_2^L).

Proof: See Hamilton and Slutsky (1990).

■

Theorem 5 says that, without further analysis, any of the three possible timing can be the endogenous outcome, that is, player 1 being leader and player 2 waiting, player 2 being leader and player 1 waiting, or the simultaneous configuration. In Theorem 6, we prove that under risk considerations, the endogenous timing outcome is sequential for both supermodular and submodular competition.

Theorem 6 *Considering either (A1) or (A2) (as in 2 or 3), and that each payoff function is monotone*[6] *increasing or decreasing as a function of the action of the other player, any of the sequential equilibria risk dominates the simultaneous one.*

Proof: Without loss of generality, we compare (x_1^L, W_2) with (x_1^N, x_2^N).

<u>Bicentric prior.</u>

- Player 2 believes that player 1 commits to x_1^L with some probability $z \in (0, 1)$ and to x_1^N with the complementary probability $1 - z$. Consequently, the best player 2 can do is to wait and play a best response. In other words, there is no action to which player 2 can commit that would imply achieving higher payoff than waiting. Therefore, the prior belief of player 1 about the behavior of player 2 is that she waits.

[6] The same monotonicity for both players.

- Player 1 believes that player 2 waits (W_2) with some probability $z \in (0,1)$, and commits to x_2^N with the complementary probability $1-z$. If player 1 waits or commits to x_1^N, she obtains Π_1^N (regardless of the value of z). On the other hand, if player 1 commits to an action $x_1(z)$ "slightly different" from x_1^N, she can achieve higher payoff. Such "slight difference" depends on the type of competition as follows:

  - If competition is supermodular and $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} > 0, i = 1, 2,$ $x_1(z)$ is above x_1^N.
  - If competition is supermodular and $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} < 0, i = 1, 2,$ $x_1(z)$ is below x_1^N.
  - If competition is submodular and $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} < 0, i = 1, 2,$ $x_1(z)$ is above x_1^N.
  - If competition is submodular and $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} > 0, i = 1, 2,$ $x_1(z)$ is below x_1^N.

  In any case, the bicentric prior of player 2 about the behavior of player 1 is that she commits.

Tracing procedure.

The starting point in the tracing procedure is the best response to the bicentric prior. Since those beliefs are that player 2 waits and player 1 commits, such starting point is precisely (x_1^L, W_2). As this is an equilibrium of the basic interaction, it is an equilibrium all the way through the procedure (for all $t \in [0,1]$). Consequently, (x_1^L, W_2) risk dominates (x_1^N, x_2^N).

■

Therefore, when extending the basic interaction with the GAC model and considering risk dominance, the endogenous timing outcome is sequential play, regardless of the type of competition and possible intrinsic differences between the firms. In other words, the sequential play is a consequence of the endogenizing game itself (and risk dominance) and not of some other characteristic of the players. However, if we want to refine among the sequential equilibria, that is, determine which one of the players is more likely of being the leader, we must base our analysis in some intrinsic difference of the players. Examples of such refinements can be found in Amir and Grilo (1999), Amir and Stepanova (2006), van Damme and Hurkens (1999) and van Damme and Hurkens (2004). We also apply that procedure in section 5.2.

4 Discussion

In this section we discuss our main results and their interpretation, considering the supermodular and submodular cases.

When competition is supermodular, both the GAC and the GOD extension models predict that the endogenous timing outcome is sequential play (Theorems 1 and 6). The intuition behind this coincidence of results becomes clearer if you recall that this is a generalization of price competition. In such cases, grossly speaking, the incentives are to wait and see what the other player does intending to undercut her price. This idea is enforced by Theorem 4, which says that the incentive to wait and see holds even when the other player sees the interaction as submodular.

When competition is submodular, results are different with each extension model. Using GAC, we obtain sequential play (Theorem 6); and using GOD, simultaneous play (Theorem Theorem 2). This submodular case can be thought as a generalization of quantity competition, and in that scenario, players want to avoid the follower position because that is the worst scenario in terms of payoff. We have found that the possibility of firms avoiding such position depends crucially on the type of market competition, i.e. the endogenizing game. In the GOD extension model, to avoid the follower position, players only need to declare "E" on the pre-play stage. Using that strategy, the least favorable scenario in terms of the payoff is to end up in simultaneous play, which occurs when the other player also chooses "E". On the other hand, when considering the GAC extension model, avoiding the follower position requires commitment, specifically, to an *aggressive* action, since we know from Hamilton and Slutsky (1990) that players do not commit to actions below the Nash Equilibrium action.[7] Therefore, in this framework avoiding the follower position is indeed risky, because if both players do so, they could end up engaged in Stackelberg warfare (both producing their leader quantity). Given that risk, they might prefer to wait and see, instead of committing.

Having noticed this fact, it is natural to ask: *if GAC and GOD could lead to very different outcomes, in which cases would be reasonable to model with GOD and in which others with GAC?* The answer of course is not trivial. Perhaps a natural approach would be related to the actual capacity of commitment of the firms. If we are dealing with firms that are strong enough to commit to *aggressive* actions, the GAC extension model seems to be appropriate. Otherwise, if they are not capable of doing that, the GOD extension model would be better, since in that case they declare an intention of being the leader (or not being a follower) *before* having to actually take an action. In any case, what extension model to use will depend

[7] When payoffs are monotone increasing in the action of the rival. When they are decreasing, players do not commit above the Nash Equilibrium.

critically on the nature of the game or market we are modeling.

Finally, a comment about Theorem 3 in the GOD case. It says that when we remove the symmetry of $\partial\Pi_i/\partial x_j$, the results are *swapped*, in the sense that we obtain simultaneous play when payoffs are supermodular, and sequential play when they are submodular. The interpretation is that the incentives to trying to avoid the follower position (in the submodular case) and "wait and see" (in the supermodular case) require knowing that the other player has the same perception about the game.

In the next section, we apply GOD and GAC to a model of differentiated quantity competition, which is something that, to the extent of our knowledge, has not been exhaustively done in the literature up to this point.

5 An illustration: differentiated quantity competition

Consider two firms, namely 1 and 2, which produce $q_1 \geq 0$ and $q_2 \geq 0$ and face linear demands

$$p_1(q_1, q_2) = A - q_1 - \alpha q_2 \qquad \text{and} \qquad p_2(q_1, q_2) = A - q_2 - \alpha q_1.$$

Where $A > 0$ is a constant that represents the size of the market and $\alpha \in (0, 1)$ is the degree of differentiation. We assume that marginal costs c_1 and c_2 are such that $c_1 < c_2$[8], and that there are no fixed costs. After their production decisions, firms receive payoffs:

$$\Pi_1(q_1, q_2) = (a_1 - q_1 - \alpha q_2)q_1 \qquad \text{and} \qquad \Pi_2(q_1, q_2) = (a_2 - q_2 - \alpha q_1)q_2.$$

Where $a_i \doteq A - c_i,\ i = 1, 2$. Note that, since $c_1 < c_2$, we have that $a_1 > a_2$.

Let us establish the equilibrium strategies and payoffs in case that competition is exogenously determined as simultaneous or sequential.

- If competition is simultaneous, equilibrium actions are:

$$q_i^N = \frac{2a_i - \alpha a_j}{4 - \alpha^2},\ i, j = 1, 2,\ i \neq j.$$

[8]Meaning that the firm 1 is more efficient. Hence, we will call it "the efficient one" sometimes throughout the paper.

And subsequently, equilibrium payoffs are:

$$\Pi_i(q_1^N, q_2^N) = \left[\frac{2a_i - \alpha a_j}{4 - \alpha^2}\right]^2, \; i, j = 1, 2, \; i \neq j. \tag{6}$$

- If competition is sequential, equilibrium actions are:

$$q_i^L = \frac{2a_i - \alpha a_j}{2(2 - \alpha^2)} \qquad \text{and} \qquad q_i^F = \frac{4a_i - \alpha^2 a_i - 2\alpha a_j}{4(2 - \alpha^2)}, \; i, j = 1, 2, \; i \neq j.$$

The respective equilibrium payoffs are:

$$\Pi_i(q_i^L, q_j^F) = \frac{[2a_i - \alpha a_j]^2}{8(2 - \alpha^2)} \text{ and } \Pi_i(q_i^F, q_j^L) = \left[\frac{4a_i - \alpha^2 a_i - 2\alpha a_j}{4(2 - \alpha^2)}\right]^2, \; i, j = 1, 2, \; i \neq j. \tag{7}$$

For simplicity, we denote $\Pi_i(x_1^N, x_2^N)$, $\Pi_i(q_1^L, q_2^F)$ and $\Pi_i(q_1^F, q_2^L)$ by Π_i^N, Π_i^L and Π_i^F $(i, j = 1, 2, \; i \neq j)$ respectively. The upper indexes L, F and N stand for leader, follower and Nash (simultaneous).

To avoid the case in which the follower firm is forced out of the market, we will be considering the following assumption.

Assumption 1

$$(4 - \alpha^2)a_2 > 2\alpha a_1.$$

In particular, this condition implies that:

$$\alpha a_1 < 2a_2.$$

In the next two subsections, we use GOD and GAC to determine which is the endogenous timing outcome in each case.

5.1 Extended Game with Observable Delay (GOD)

The normal form of the extended game can be represented in Table 4, where the payoffs are those defined in (6) and (7).

	E	L
E	Π_i^N, Π_j^N	Π_i^L, Π_j^F
L	Π_i^F, Π_j^L	Π_i^N, Π_j^N

Table 4: Extended Game with Observable Delay.

The first result says that both firms choose *early* in the pre-play stage, which means simultaneous competition in the basic interaction.

Proposition 1 *The action profile* (E, E) *is the unique Nash Equilibrium of the reduced game in Table 4.*

Proof:

Direct from Assumption 1 and that $a_1 > a_2$.

■

Note that, if $c_1 = c_2 = c$, the result remains true. Therefore, regardless of the cost difference, the unique SPE of the extended game is both firms choosing E in the pre-play stage, which induces simultaneous play in the basic interaction. In particular, we can conclude that the endogenous timing result does not depend on the intrinsic difference of the firms, nor the magnitude of the degree of differentiation parameter. More specifically, α is only relevant for its sign not its magnitude. As predicted by Theorem 2, the underlying fact ruling the outcome is related to the qualitative (not quantitative) behavior of the best responses.

5.2 Extended Game with Action Commitment (GAC)

Let us denote by $S^1 = (q_1^L, W_2)$, $S^2 = (W_1, q_2^L)$, and $N = (q_1^N, q_2^N)$ the three profiles supported by SPE of the extended game.

Proposition 2 *Any of the Stackelberg equilibria risk dominates the simultaneous one, i.e.* S^1 *and* S^2 *risk dominate* N*.*

Proof: Without loss of generality, let us compare S^1 and N.

Bicentric prior.

- Firm 2 believes that firm 1 plays q_1^L with probability z_2 and q_1^N with probability $1-z_2$. Consequently, the best firm 2 can do for all $z_2 \in (0,1)$ is to wait. Therefore, the prior belief of firm 1 is that 2 will certainly wait.

- Firm 1 thinks that 2 will play W_2 with probability z_1 and q_2^N with probability $1-z_1$. If firm 1 waits, her payoff is Π_1^N (for all $z_1 \in (0,1)$). However, for $z_1 > 0$, firm 1 can achieve a higher expected payoff by to:

$$q_1(z_1) = \frac{a_1}{(2 - z_1\alpha^2)} - \frac{\alpha(1-z_1)(2a_2 - \alpha a_1)}{(4-\alpha^2)(2-z_1\alpha^2)} - \frac{\alpha z_1 a_2}{2(2-z_1\alpha^2)}.$$

Observation 3 *Note that:*

- $q_1(z_1)$ *is an increasing function of* z_1*. This is, the optimal commitment quantity of player* 1 *is bigger as player* 2 *is more likely to wait.*

- *For all* $z_1 \in (0,1)$*, we have that* $q_1(z_1) > q_1^N$*. Also, the payoff obtained by playing* $q_1(z_1)$ *against the mixed strategy of waiting with probability* z_1 *and committing to* q_2^N *with* $1-z_1$ *is greater than* Π_1^N*.*

- $q_1(z_1 = 1) = q_1^L$*, i.e. if firm* 1 *knows that her rival will wait, the best option is to commit to the leader equilibrium quantity. On the other hand,* $q_1(z_1 = 0) = q_1^F$*.*[9]

Tracing procedure.

The starting point ($t = 0$) is defined by the best response to the prior belief. In this case, such best response is that:

- Firm 1 commits to q_1^L.

- Firm 2 waits.

Therefore, in $t = 0$ the unique equilibrium is S^1. As S^1 is an equilibrium of the original game, it is also an equilibrium $\forall t \in [0,1]$ in the tracing procedure. Consequently, S^1 is the risk dominant equilibrium.

■

[9]However, from Hamilton and Slutsky (1990), we know that the players do not commit to quantities below the Nash equilibrium. In such case, it is better to *wait*.

As in the GOD case, this result remains true if $c_1 = c_2 = c$. The fact of competition being sequential with the GAC extension model does not depend on the firms having an intrinsic difference.

The next step is to compare the sequential equilibria and the essential element to do so is precisely the cost difference ($c_1 < c_2$), since it will allow the discrimination based on the risk dominance idea. In Proposition 3, we will show that the equilibrium in which the efficient firm is the leader (S^1) risk dominates the one in which the inefficient has the leadership (S^2).

Before proceeding with such analysis, let us make a point about the notation. As the original payoff functions of our model are defined over $\mathbb{R}_+^2$, they technically cannot be evaluated in a mixed strategy which entails waiting. We introduce the auxiliary payoff functions $u_i(\cdot, \cdot),\ i, j = 1, 2,\ i \neq j$, that satisfy the following:

- $u_i(q_i, q_j) = \Pi_i(q_i, q_j)\ \forall q_i, q_j \in \mathbb{R}_+^2$.
- $u_i(q_i, W_j) = \Pi_i(q_i, q_j(q_i))\ \forall q_i \in \mathbb{R}_+$, where $q_j(\cdot)$ is the best response of player j.
- $u_i(W_i, q_j) = \Pi_i(q_i(q_j), q_j)\ \forall q_j \in \mathbb{R}_+$, where $q_i(\cdot)$ is the best response of player i.
- $u_i(W_i, W_j) = \Pi_i^N$.
- $u_i(q_i, zq_j + (1-z)W_j) = zu_i(q_i, q_j) + (1-z)u_i(q_i, W_j),\ \forall z \in [0, 1]$ and $q_j \in \mathbb{R}_+$.
- $u_i(W_i, zq_j + (1-z)W_j) = zu_i(W_i, q_j) + (1-z)u_i(W_i, W_j),\ \forall z \in [0, 1]$ and $q_j \in \mathbb{R}_+$.

Proposition 3 *If $c_1 < c_2$, then $S^1 = (q_1^L, W_2)$ risk dominates $S^2 = (W_1, q_2^L)$.*

Proof: Let us start by specifying the bicentric prior and then we move on to the tracing procedure.

Bicentric prior.

Firm 2 believes that 1 will commit to q_1^L with some probability z and wait with the complementary probability $1 - z$. If firm 2 commits to a quantity q_2, her expected payoff is:

$$u_2(zq_1^L + (1-z)W_1, q_2) = zu_2(q_1^L, q_2) + (1-z)u_2(W_1, q_2).$$

The optimal commitment is:

$$q_2^*(z) = a_2 \left[\frac{4 - \alpha^2(2-z)}{2(2-\alpha^2)(2-\alpha^2(1-z))} \right] - a_1 \left[\frac{2\alpha - \alpha^3 + z\alpha^3}{2(2-\alpha^2)(2-\alpha^2(1-z))} \right]. \tag{8}$$

And the payoff due to this optimal commitment found in (8) is:

$$u_2(zq_1^L + (1-z)W_1, q_2^*(z)) = \frac{\left[a_2(4-\alpha^2(2-z)) - \alpha a_1(2-\alpha^2(1-z))\right]^2}{8(2-\alpha^2)^2(2-\alpha^2(1-z))}.$$

Note that, if $z = 1$, the best firm 2 can do is to wait, and if $z = 0$, the best she can do is to commit to her leader equilibrium quantity. Naturally, the best response of firm 2 will have a threshold structure. Committing is better than waiting if

$$u_2(zq_1^L + (1-z)W_1, q_2^*(z)) \geq u_2(zq_1^L + (1-z)W_1, W_2).$$

Which holds if:

$$z \leq z_2(\alpha) \doteq \frac{2(2a_2 - \alpha a_1)^2(2-\alpha^2)}{8\alpha^3 a_1 a_2 - 2\alpha^4 a_1^2 + a_2^2(16 - 16\alpha^2 + \alpha^4)}.$$

Therefore, the best response of 2 to player 1 mixed strategy of playing q_1^L with probability z and W_1 with probability $1-z$ is given by:

$$b_2(z) = \begin{cases} W_2 & \text{if } \ z > z_2(\alpha). \\ q_2^*(z) & \text{if } \ z \leq z_2(\alpha). \end{cases}$$

We denote $z_2(\alpha)$ by z_2. The procedure for the other player is analogous and it is possible to show that $0 < z_2 < z_1$ (for all α), which means that it is more likely for the efficient firm to commit.

To summarize, the prior belief of firm 1 about firm 2's behaviour is that she will use the following mixed strategy:

$$m_2 \doteq b_2(Z), \ Z \sim U([0,1]). \tag{9}$$

The only relevant characteristics (for posterior calculations) of a bicentric prior are the probability to wait, the mean of the commitment quantity and its variance. For player 2, these characteristics are denoted $w_2,\ \mu_2$ and ν_2.

$$w_2 \doteq \mathbb{P}(\text{waiting}) = 1 - z_2.$$

$$\mu_2 \doteq \mathbb{E}(q_2^*(z)) = \frac{a_2 - \alpha a_1}{2(2-\alpha^2)} + \frac{a_2}{2\alpha^2 z_2} \cdot \ln\left(\frac{2-\alpha^2+\alpha^2 z_2}{2-\alpha^2}\right).$$

$$\nu_2 \doteq \mathbb{V}(q_2^*(z)) = \frac{a_2^2(1-\alpha^2+\alpha^2 z_2)}{4\alpha^2 z_2(2-\alpha^2+\alpha^2 z_2)} - \frac{a_2^2}{4\alpha^4 z_2^2}\ln^2\left(\frac{2-\alpha^2+\alpha^2 z_2}{2-\alpha^2}\right).$$

Analogous for player 1.

Tracing procedure.

Let us define the ratio between a_1 and a_2, as our results will depend on such ration and α.

$$d \doteq \frac{a_1}{a_2}.$$

From assumption 1, we know that $1 < d < 1.5$.

In Lemma 2 and Lemma 3, we give the parameter range for which each player decides to commit and to wait when the tracing procedure starts. Both proofs can be found in the Appendix.

Lemma 2 *If $d + \frac{5000}{1387}\alpha \geq \frac{12611}{2774}$, the efficient player prefers committing over waiting at the beginning of the tracing procedure. That is, if $d + \frac{5000}{1387}\alpha \geq \frac{12611}{2774}$, then:*

$$u_1(W_1, m_2) < \max_{q_1} u_1(q_1, m_2).$$

If the opposite holds, player 1 prefers waiting when the procedure starts.

Lemma 3 *If $d - \frac{800}{163}\alpha \leq \frac{15599}{4075}$, the inefficient player prefers committing over waiting at the beginning of the tracing procedure. That is, if $d - \frac{800}{163}\alpha \leq \frac{15599}{4075}$, then:*

$$u_2(m_1, W_2) < \max_{q_2} u_2(m_1, q_2).$$

If the opposite holds, player 2 prefers waiting when the procedure starts.

Given these two results, we can understand what players will do at the beginning of the tracing procedure for every value of d and α. In Figure 1, we see three regions[10]: blue, red,

[10]The proportions are obviously not precise. The purpose is just to provide intuition.

and green. In the blue zone, both players start the tracing procedure waiting; in the green one, the efficient player commits and the inefficient one waits; and finally, in the red region, both players start the tracing procedure committing. We divide the analysis into three cases, one for each region.

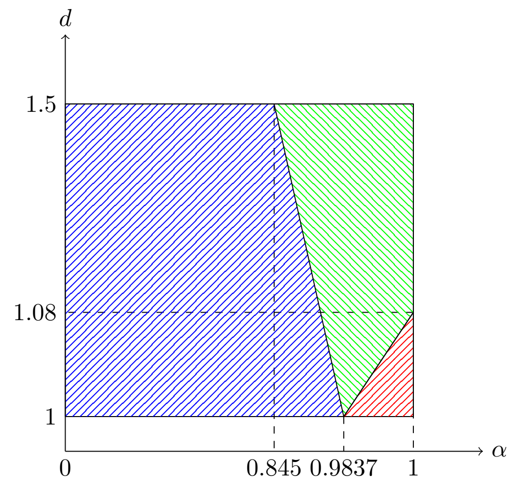

Figure 1: Cases in the parameter space.

Case 1 (Green region).

Player 1 starts the procedure committing and player 2 starts waiting. The desired result is proved.

Case 2 (Red region).

Both players start the procedure committing. Denote the optimal commitment quantities at each $t \in [0,1]$ by q_1^t and q_2^t. We define the gains for committing versus waiting (at each moment of the procedure as), considering that the other player is committing:

$$\varphi_1(t) \doteq u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t) \qquad \text{and} \qquad \varphi_2(t) \doteq u_2^t(q_1^t, q_2^t) - u_2^t(q_1^t, W_2).$$

Where $u_i^t(\cdot,\cdot), i = 1,2$ are the payoffs at $t \in [0,1]$ as defined in (1).

Lemma 4 *Let be $s_t = (s_1^t, s_2^t)$ the equilibrium on time t in the tracing procedure. Then, there exist $i \in \{1,2\}$ and $t < 1$, such that $s_i^t = W_i$.*

Lemma 4 says that is not possible that both firms keep on committing up to the end of the tracing procedure (at least one must switch to *wait* before the end). The next result establishes that it must be the inefficient player the first one to do so.

Lemma 5 *For $i \in \{1,2\}$, let t_i be the last point in time for which it is convenient for firm i to commit. This is, if we define*

$$t_i \doteq \sup\{\tau \in [0,1] : \ \varphi_i(t) \geq 0 \ \forall t \in [0,\tau]\}.$$

Then, $t_1 > t_2$.

Proofs of Lemma 4 and Lemma 5 can be found in the Appendix.

Therefore, we know that if both firms start the tracing procedure committing optimally, the inefficient player is the first one to become indifferent between committing and waiting. The next step is to describe how the procedure evolves after reaching t_2.

In Figure 2 we can see the optimal actions for the efficient player as a function of time in the tracing procedure when the inefficient player is waiting for sure (blue) and when is committing for sure (red)[11]. Denote by $q_1^C(t)$ and $q_2^C(t)$ the optimal commitment quantities against each other.

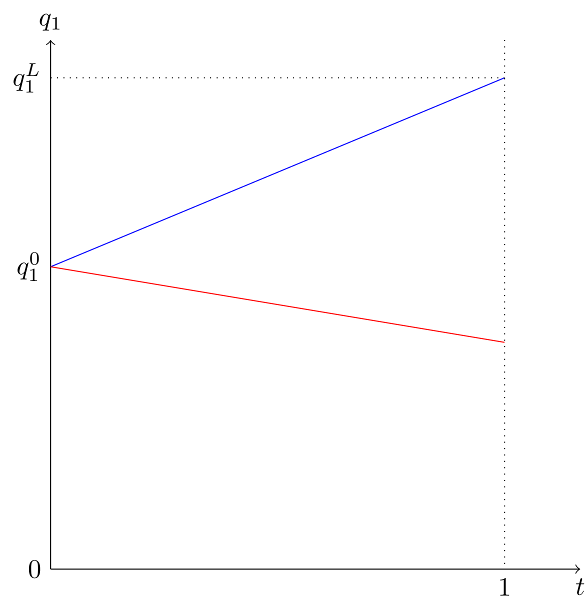

Figure 2: Tracing procedure actions for the efficient player.

We know, from lemmas 4 and 5, that the efficient player starts the tracing procedure moving along the red curve up to point A, as seen in Figure 3. From that moment, it makes

[11] Such optimal actions are not necessarily straight lines. However, the purpose of the Figure is only to provide intuition.

no sense to keep along such red curve because now player 2 is indifferent between waiting and committing. In particular, player 1 can now commit to bigger quantities, as long as such quantities keep the indifference condition over player 2.

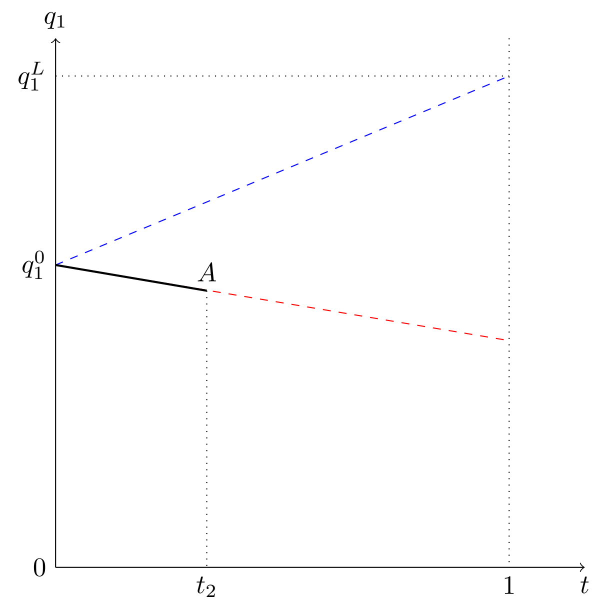

Figure 3: First stage of the procedure.

Denote by $q_1^I(t)$ the curve of quantities that leave player 2 indifferent between waiting and committing optimally. It must be the case that $q_1^I(t)$ intersects the red line at that point A and the tracing procedure must continue along this $q_1^I(t)$ for some time, with the equilibrium being with player 1 committing to $q_1^I(t)$ and player 2 mixing between waiting (with some probability) and committing to some $q_2^I(t)$ (with the complementary probability). These optimal actions and probabilities can be determined by optimality and indifference conditions but are not necessary to make the argument. Note first that player 2 strictly prefers committing (over waiting) when player 1 commits to $q_1^C(t)$ for all $t < t_2$. Furthermore, we know that committing is less attractive when the commitment action of the other player is bigger. Therefore, $q_1^I(t)$ necessarily intersects the curve $q_1^C(t)$ from above. Consequently, the proof, in this case, will be complete if we can show that $q_1^I(t)$ bends backward at point A, since this fact would imply that $q_1^I(t)$ necessarily intersects the blue line at some $t < t_2$, and the tracing procedure continues along the such blue line until the desired equilibrium. In summary, we want to prove that the setting is as shown in Figure 4, that is, the procedure starts with the efficient player moving along the red curve up to point A, then it moves along the green curve up to point B, and finally continues along the blue line until the desired equilibrium.

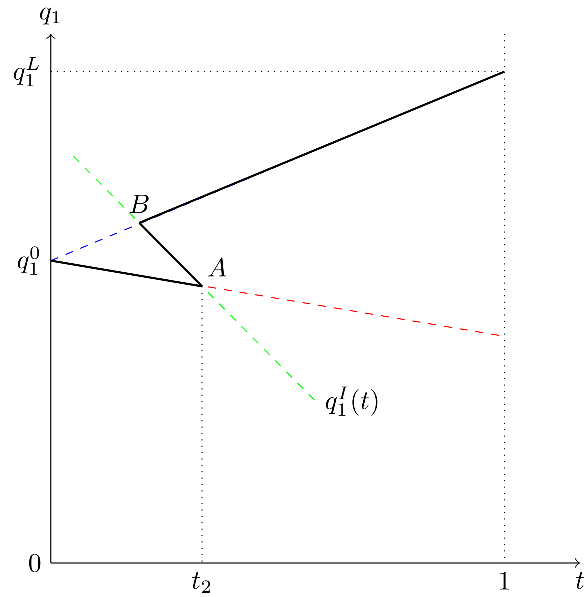

Figure 4: Final stage of the tracing procedure.

To do so, Note first that the best responses against the commitment of the other player are given by:

$$q_i^t(q_j) = \frac{(1-t)\left[a_i - \frac{\alpha a_j}{2} + \alpha z_j\left(\frac{a_j}{2} - \mu_j\right)\right] + t(a_i - \alpha q_j)}{2(1-t)z_j + (1-t)(1-z_j)(2-\alpha^2) + 2t}, \ i, j = 1, 2, \ i \neq j.$$

Recalling that $q_1^C(t)$ and $q_2^C(t)$, are the optimal commitment quantities against each other, we can re write the previous expressions as:

$$q_i^t(q_j) = \frac{-t(a_i - \alpha q_j^C(t)) + [2(1-t)z_j + (1-t)(1-z_j)(2-\alpha^2) + 2t]\, q_i^C(t) + t(a_i - \alpha q_j)}{2(1-t)z_j + (1-t)(1-z_j)(2-\alpha^2) + 2t}. \tag{10}$$

For $i, j = 1, 2$, $i \neq j$.

We know that $q_1^I(t)$ must be the best response against player 2's strategy of waiting with some probability and committing to $q_2^I(t)$ with the complementary probability. Therefore, since the optimal commitment strategy is increasing in the probability of waiting, we have that $q_1^I(t) \geq q_1^t(q_2^I(t))$. Using (10), this is equivalent to:

$$\begin{aligned}&\left[2(1-t)z_2+(1-t)(1-z_2)(2-\alpha^2)+2t\right]q_1^I(t)\\ \geq -t(a_1-\alpha q_2^C(t)) &+ \left[2(1-t)z_2+(1-t)(1-z_2)(2-\alpha^2)+2t\right]q_1^C(t)+t(a_1-\alpha q_2^I(t))\end{aligned}$$

Multiplying both sides by $[2(1-t)z_2+(1-t)(1-z_2)(2-\alpha^2)+2t]$ and using (10) again but for player 2, we obtain that:

$$\begin{aligned}&\left(\frac{[2(1-t)z_2+(1-t)(1-z_2)(2-\alpha^2)+2t][2(1-t)z_1+(1-t)(1-z_1)(2-\alpha^2)+2t]-\alpha^2t^2}{2(1-t)z_1+(1-t)(1-z_1)(2-\alpha^2)+2t}\right)q_1^I(t)\\ \geq &\left(\frac{[2(1-t)z_2+(1-t)(1-z_2)(2-\alpha^2)+2t][2(1-t)z_1+(1-t)(1-z_1)(2-\alpha^2)+2t]-\alpha^2t^2}{2(1-t)z_1+(1-t)(1-z_1)(2-\alpha^2)+2t}\right)q_1^C(t).\end{aligned}$$

Which, in turn, implies that $q_1^I(t) \geq q_1^C(t)$. This means that the optimal commitment of the efficient firm bends backwards in t_2 and consequently, proves the result.

Case 3 (Blue region).

In this case, both players start the tracing procedure waiting. Let us first define the gains of committing versus waiting for each player, when the other player is waiting.

$$\begin{aligned}\phi_1(t) &\doteq u_1^t(q_1^t, W_2) - u_1^t(W_1, W_2)\\ &= (1-t)u_1(q_1^t, m_2) + tu_1(q_1^t, W_2) - (1-t)u_1(W_1, m_2) - tu_1(W_1, W_2).\\ \phi_2(t) &\doteq u_2^t(W_1, q_2^t) - u_2^t(W_1, W_2)\\ &= (1-t)u_2(m_1, q_2^t) + tu_2(W_1, q_2^t) - (1-t)u_2(m_1, W_2) - tu_2(W_1, W_2).\end{aligned}$$

Where

$$q_i^t = \frac{2(1-t)z_j\,(a_i-\alpha\mu_j)+(1-z_j+tz_j)\,(2a_i-\alpha a_j)}{2\,(2-\alpha^2+\alpha^2 z_j-\alpha^2 t z_j)},\ \ i,j=1,2,\ \ i\neq j \qquad (11)$$

are the are the optimal commitment actions at moment t of the tracing procedure, when the other player is waiting.[12]

[12] These are not the same q_1^t and q_2^t of Case 2 previously analysed.

In Lemma 6, we show that both players must switch from *waiting* to committing before the end of the tracing procedure; and in Lemma 7 that is the efficient player the first one to do so. Both proofs can be found in the Appendix.

Lemma 6 *Let be $s_t = (s_1^t, s_2^t)$ the equilibrium on time t in the tracing procedure. Then, there exists t_i, such that $s_i^t = q_i^t$ $\forall t \geq t_i$, $i = 1, 2$.*

Lemma 7 *For $i \in \{1, 2\}$, let t_i be the last point in time for which it is convenient for firm i to wait. This is,*

$$t_i \doteq \sup\{\tau \in [0, 1] : \ \phi_i(t) \leq 0 \ \forall t \in [0, \tau]\}.$$

Then, $t_1 < t_2$.

At t_1 the efficient player will start committing according to (11) and player 2 will continue waiting. The process continues this way until $t = 1$, and we know from (11) that, at such point:

$$q_1^1 = \frac{2a_1 - \alpha a_2}{2(2 - \alpha^2)} = q_1^L.$$

Which proves the result in this case, and the proof of Proposition 3 is completed.

■

Proposition 3 shows that the differentiation parameter plays no role in the result when using the GAC extension model, since we are obtaining the analogous to that in van Damme and Hurkens (1999): efficient player emerges as leader. This fact enforces the idea the endogenous timing results are primary driven by the qualitative (and not quantitative) behaviour of the reaction curves.

6 Conclusions

In this article we challenged the commonly accepted idea that the answer to the "Stackelberg or Cournot" question is primarily determined by an intrinsic difference, this is, some characteristic that makes firms different, such as marginal cost, capacity of production, or quality. We proved that the endogenous timing outcomes depend crucially on the "endogenizing game" and also on the type of competition (supermodular or submodular). Specifically, we considered a general model of duopolistic competition as a basic interaction, extended it using the Game with Action Commitment and the Game with Observable Delay from Hamilton and Slutsky (1990), and used the risk dominance concept from Harsanyi et al. (1988) to refine in the presence of multiple equilibria.

In the case of supermodular competition, we found that both extension models lead to the same endogenous timing outcome, in the sense that they predict that competition will result in sequential play. The idea is that, when competition is supermodular, players naturally prefer to see what the other does before choosing their own action because that allows them to "undercut" (if we are thinking of price competition) the action of the other player. On the other hand, when competition is submodular, GOD and GAC lead to very different results. In particular, when the extension model is GOD, the endogenous timing outcome is simultaneous competition; and when the extension is using GAC, competition is sequential under risk considerations. Our interpretation is that, in this basic interaction, players want to avoid the follower position and trying to materialize that preference might entail more or less risk depending on the endogenizing game. In the GOD doing so has no risk whatsoever because firms just need to declare "E" in the pre-play stage, and by doing so, the *worst* scenario they can face is simultaneous competition. Such incentives lead to simultaneous play in the basic interaction. In the GAC extension model, they need to commit to a specific action and, even more, it must be a *risky* action.[13] To avoid such risk, firms might prefer to wait and see what the rival does, which induces sequential play in the basic interaction.

The findings we present, that is, that the extension models give "different" results (sequential versus simultaneous) in the submodular case and the "same" (sequential) result in the supermodular case is a generalization and ultimately a formalization of something one can observe in the literature. In the case of price competition with differentiated goods, you can see in Amir and Stepanova (2006) that GOD leads to sequential play, and the same result in terms of the endogenous timing outcome can be found in van Damme and Hurkens (2004) using GAC. In the case of quantity competition with homogeneous goods, Amir and Grilo (1999) obtain simultaneous play using GOD, and van Damme and Hurkens (1999) obtain sequential play using GAC.

In summary, based on our results, we can completely determine the endogenous timing outcome of a model, only knowing the endogenizing game and the shape of the best response functions. This characterization is presented in Table 5.

[13] Above or below the Nash equilibrium action depending on whether the payoffs are increasing or decreasing in the action of the other player.

	Supermodular competition	**Submodular competition**
GOD	Sequential	Simultaneous
GAC	Sequential	Sequential

Table 5: Characterization of endogenous timing outcomes.

Finally, we considered the particular case of differentiated quantity competition with different marginal costs to show not only that the general result for submodular competition holds in that particular setting but essentially to prove that, among the two sequential equilibria, the risk dominant is the one in which the efficient player becomes the leader. The interpretation is that committing is riskier for the inefficient player and therefore she prefers to wait. With this model we are showing that the differentiation parameter plays no role in the endogenous timing results when using the GAC extension model and risk dominance as a refinement criterion, in the sense that we obtained the same outcome of van Damme and Hurkens (1999), where $\alpha = 1$. This result highlights our main idea that the endogenous timing results depend on the reaction functions only through their qualitative (and not quantitative) behavior.

Appendix

Lemma 1.

Proof: Let us analyze first the condition to have a first mover advantage. Consider the following for player $i = 1, 2$, $j \neq i$:

$$
\begin{aligned}
\Pi_i(x_i^L, x_j^F) &> \Pi_i(x_i^N, x_j^N) \\
&\geq \Pi_i(x_i^F, x_j^N) \\
&> \Pi_i(x_i^F, x_j^L).
\end{aligned}
$$

The first inequality is trivial, the second one is the definition of Nash equilibrium, and the last one is true under either assumption 1.1 or 2.1. The result follows.

Now, let us analyze the case in which the player has a second mover advantage. Consider the following inequalities:

$$
\begin{aligned}
\Pi_i(x_i^F, x_j^L) &\geq \Pi_i(x_i^L, x_j^L) \\
&> \Pi_i(x_i^L, x_j^F).
\end{aligned}
$$

In this case, the first inequality is by definition of best response, and the second one is true when you consider assumption 1.2 or 2.2. ■

Theorem 3.

Proof: Let us assume, without loss of generality, that

$$\frac{\partial \Pi_1(x_1, x_2)}{\partial x_2} > 0 \text{ and } \frac{\partial \Pi_2(x_1, x_2)}{\partial x_1} < 0.$$

Once again, recall that (4) and (5) are valid for both the supermodular and submodular case. Given the monotonicity of the payoffs, it is possible to show the following:

$$x_2^F > x_2^N \text{ and } x_1^F < x_1^N. \tag{12}$$

Now, we divide the analysis for each assumption.

- Let us assume the competition is supermodular, therefore best responses are increasing. Thus, from (12), we have:

$$x_1^F < x_1^N < x_1^L \text{ and } x_2^L < x_2^N < x_2^F.$$

Then, we know from Lemma 1 that both players have a first mover advantage. The result follows.

- Consider now that competition is submodular, which implies that best response functions are decreasing. From (12), we have that:

$$x_1^L, x_1^F < x_1^N \text{ and } x_2^L, x_2^F > x_2^N.$$

Let us analyze the three possible cases, given the shape of the best response functions.

(i) If $x_1^L < x_1^F$ and $x_2^L > x_2^F$, we know from Lemma 1 that both players have a second mover advantage.

(ii) If $x_1^L < x_1^F$ and $x_2^L < x_2^F$, player 2 has a second mover advantage.

(iii) If $x_1^L > x_1^F$ and $x_2^L > x_2^F$, player 1 has a second mover advantage.

■

Theorem 4.

Proof: We start by considering (4) and (5), which are valid regardless of the super or sub modularity.

- Let us assume first that $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} > 0,\ i = 1, 2$. In this case, we have:

$$x_2^F > x_2^N \text{ and } x_1^F > x_1^N.$$

Considering the supermodularity of player 1 and submodularity of player 2, we obtain:

$$x_1^L < x_1^N < x_1^F \text{ and } x_2^F, x_2^L > x_2^N.$$

We inmmediately have, from Lemma 1, that player 2 has a first mover advantage. Now, since in particular $x_2^N < x_2^L$, the following holds:

$$\Pi_1(x_1^N, x_2^N) < \Pi_1(x_1^N, x_2^L) < \Pi_1(x_1^F, x_2^L).$$

Which implies that the endogenous timing outcome is (L, E), this is, sequential play with player 2 being leader.

- Let us assume now that $\dfrac{\partial \Pi_i(x_i, x_j)}{\partial x_j} < 0,\ i = 1, 2..$ In this case:

$$x_1^L > x_1^N > x_1^F \text{ and } x_2^L, x_2^F < x_2^N.$$

The rest is analogous to the previous scenario.

■

<u>Lemma 2.</u>

Proof: Let us start by obtaining the optimal commitment quantity of player 1 at the beginning of the tracing procedure ($t = 0$), when the other player is also committing:

$$q_1^* = \frac{a_1 - \dfrac{\alpha a_2}{2} + z_2\left(\dfrac{\alpha a_2}{2} - \alpha\mu_2\right)}{2 - \alpha^2 + \alpha^2 z_2}.$$

The payoff using this strategy is:

$$\begin{aligned} u_1(q_1^*, m_2) &= z_2(a_1 - q_2^* - \alpha\mu_2)q_1^* + (1 - z_2)\left(\frac{2a_1 - q_1^*(2 - \alpha^2) - \alpha a_2}{2}\right) q_1^* \\ &= \frac{(2a_1 + \alpha(a_2(z_2 - 1) - 2z_2\mu_j))^2}{8(2 - \alpha^2(1 - z_2))}. \end{aligned}$$

On the other hand, if firm 1 waits, her payoff is:

$$u_1(W_1, m_2) = z_2 \left(\frac{(a_1 - \alpha\mu_2)^2}{4} + \frac{\alpha^2 \nu_2}{4} \right) + (1 - z_2) \left(\frac{2a_1 - \alpha a_2}{4 - \alpha^2} \right)^2.$$

We have to verify under which conditions it is true that $u_1(q_1^*, m_2) - u_1(W_1, m_2) \geq 0$. Up to a negative scalar, such difference is equal to:

$$\begin{aligned}
&= \frac{2K\left(\alpha^3(\alpha d - 1)(\alpha - 2d)\left((\alpha^2 - 8) + 2\alpha d\right) - \left(-2\alpha^4 d^2 + 8\alpha^3 d + (\alpha^4 - 16\alpha^2 + 16)\right)K\right)}{\alpha^2} \\
&+ \frac{-8\alpha^8 d^6 + 80\alpha^7 d^5 + 2\left(\alpha^4 - 10\alpha^2 - 132\right)\alpha^6 d^4 + 2\left(\alpha^6 + 5\alpha^4 - 40\alpha^2 - 208\right)\alpha^4 d^2}{-2\alpha^4 d^2 + 8\alpha^3 d + (\alpha^4 - 16\alpha^2 + 16)} \\
&+ \frac{8\left(-7\alpha^4 + 24\alpha^2 + 16\right)\alpha^3 d + \left(-3\alpha^{10} + 41\alpha^8 - 168\alpha^6 + 352\alpha^4 - 512\alpha^2 + 256\right) - 12\left(\alpha^4 - 8\alpha^2 - 32\right)\alpha^5 d^3}{-2\alpha^4 d^2 + 8\alpha^3 d + (\alpha^4 - 16\alpha^2 + 16)}
\end{aligned}$$

Where $K = \log\left[\frac{\left(\alpha^2 - 4\right)^2}{-2\alpha^4 d^2 + 8\alpha^3 d + (\alpha^4 - 16\alpha^2 + 16)}\right]$.

Therefore, if we want $u_1(q_1^*, m_2) - u_1(W_1, m_2)$ to be positive we can check the conditions for which the previous expression is negative. In Figure 5, we can see the plotting of the expression (orange) and the constant function equal to zero (blue).

The line that defines the region for which the expression is negative goes from $(\alpha = 0.845, d = 1.5)$ to $(\alpha = 0.9837, d = 1)$. Therefore, the desired results holds for:

$$d + \frac{5000}{1387}\alpha \geq \frac{12611}{2774}.$$

■

Lemma 3.

Proof: We must check the conditions under which

$$u_2(m_1, W_2) < \max_{q_2} u_2(m_1, q_2).$$

Proceeding analogously as in the proof of Lemma 2, the result will hold if the following expression is negative.

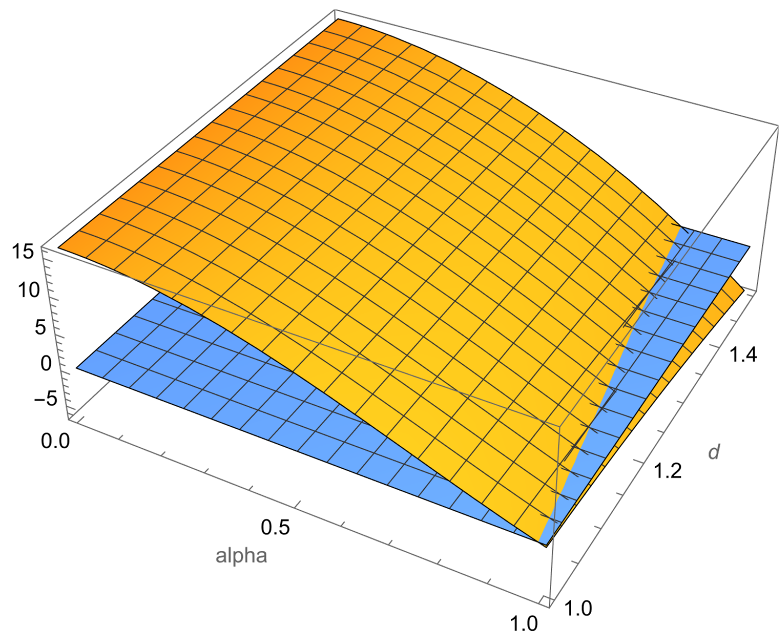

Figure 5: Lemma 2.

$$
\begin{aligned}
&-\frac{2d^2\left(-2\alpha^4+8\alpha^3 d+\left(\alpha^4-16\alpha^2+16\right)d^2\right)\log^2\left[\frac{\left(\alpha^2-4\right)^2 d^2}{-2\alpha^4+8\alpha^3 d+\left(\alpha^4-16\alpha^2+16\right)d^2}\right]}{\alpha^2}\\
&-2\alpha d(\alpha d-2)(d-\alpha)\left(\left(\alpha^2-8\right)d+2\alpha\right)\log\left[\frac{\left(\alpha^2-4\right)^2 d^2}{-2\alpha^4+8\alpha^3 d+\left(\alpha^4-16\alpha^2+16\right)d^2}\right]\\
&+\frac{-8\alpha^8+80\alpha^7 d+2\left(\alpha^6+5\alpha^4-40\alpha^2-208\right)\alpha^4 d^4+2\left(\alpha^4-10\alpha^2-132\right)\alpha^6 d^2-12\left(\alpha^4-8\alpha^2-32\right)\alpha^5 d^3}{-2\alpha^4+8\alpha^3 d+\left(\alpha^4-16\alpha^2+16\right)d^2}\\
&+\frac{8\left(-7\alpha^4+24\alpha^2+16\right)\alpha^3 d^5+\left(-3\alpha^{10}+41\alpha^8-168\alpha^6+352\alpha^4-512\alpha^2+256\right)d^6}{-2\alpha^4+8\alpha^3 d+\left(\alpha^4-16\alpha^2+16\right)d^2}.
\end{aligned}
$$

The plot of the previous expression can be seen in Figure 6.

The line that defines the region for which the expression is negative goes from ($\alpha = 0.9837, d = 1$) to ($\alpha = 1, d = 1.08$). Therefore, the desired results holds for:

$$d - \frac{800}{163}\alpha \leq \frac{15599}{4075}.$$

■

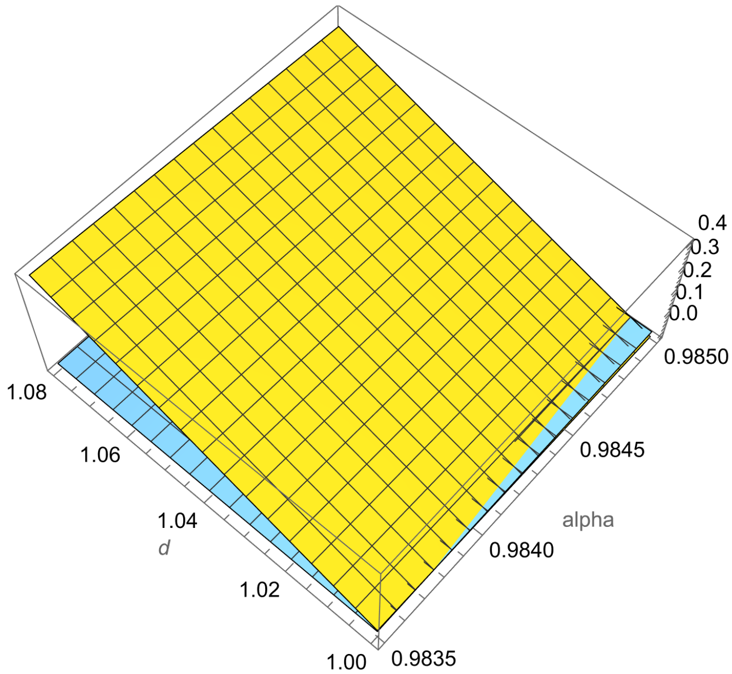

Figure 6: Lemma 3.

Lemma 4.

Proof: Reasoning by contradiction, let us assume that both players find it optimal to keep committing up to the end of the procedure. The strategy to prove the result will be to show that, for player 1, the gain of committing versus waiting is negative for some $t < 1$ (it is analogous for player 2).

Observation 4 *Recall that*

$$\varphi_1(t) \doteq u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t),$$

where

$$u_1^t(q_1^t, q_2^t) = (1-t)u_1(q_1^t, m_2) + tu_1(q_1^t, q_2^t).$$

Since at $t = 1$ the payoff functions coincide with those of the original game, it follows that $q_1^1 = q_1^N$. Evaluating in $t = 1$, we have:

$$\begin{aligned}\varphi_1(1) &= u_1^1(q_1^1, q_2^1) - u_1^1(W_1, q_2^1)\\ &= u_1(q_1^N, q_2^N) - u_1(W_1, q_2^N)\\ &= 0.\end{aligned}$$

Therefore, there is no gain associated to commit versus waiting at time $t = 1$. We will prove that $\varphi_1'(1) > 0$, which implies that $\varphi_1(t) < 0$ for some $t < 1$. The derivative of $\varphi_1(t)$ is given by:

$$\begin{aligned}\varphi_1'(t) = &\frac{\partial[u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t)]}{\partial t} + \frac{\partial[u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t)]}{\partial q_2} \cdot \frac{\partial q_2^t}{\partial t} + \underbrace{\frac{\partial[u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t)]}{\partial q_1}}_{=0} \cdot \frac{\partial q_1^t}{\partial t}\\ = &\underbrace{\frac{\partial[u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t)]}{\partial t}}_{\doteq A} + \underbrace{\frac{\partial[u_1^t(q_1^t, q_2^t) - u_1^t(W_1, q_2^t)]}{\partial q_2} \cdot \frac{\partial q_2^t}{\partial t}}_{\doteq B}.\end{aligned}$$

Let us analyse both of the terms (A and B) in the previous expression when $t = 1$.

$$A = \underbrace{u_1(q_1^N, q_2^N) - u_1(W_1, q_2^N)}_{=0} - u_1(q_1^N, m_2) + u_1(W_1, m_2) = u_1(W_1, m_2) - u_1(q_1^N, m_2) > 0.$$

The second term is:

$$\begin{aligned}B &= \left[t \cdot \frac{\partial u_1(q_1^t, q_2^t)}{\partial q_2} - t \cdot \frac{\partial u_1(W_1, q_2^t)}{\partial q_2}\right] \cdot \frac{\partial q_2^t}{\partial t}\\ &= \left[-\alpha t q_1^t - t \cdot \frac{1}{4} \cdot [2\alpha^2 q_2^t - 2\alpha a_1]\right] \cdot \frac{\partial q_2^t}{\partial t}.\end{aligned}$$

Evaluating in $t = 1$, we obtain:

$$\left[-\alpha q_1^N + \frac{1}{2} \cdot [\alpha a_1 - \alpha^2 q_2^N]\right] \cdot \frac{\partial q_2^t}{\partial t} = \left[-\alpha \cdot \frac{2a_1 - \alpha a_2}{4 - \alpha^2} + \frac{1}{2}\left[\alpha a_1 - \alpha^2 \cdot \frac{2a_2 - \alpha a_1}{4 - \alpha^2}\right]\right] \cdot \frac{\partial q_2^t}{\partial t} = 0.$$

Therefore, we have the desired result. ■

Lemma 5.

Proof:

After some algebra, it is possible to show that:

$$\varphi_i(t) = (1-t)\left[z_j(a_i - q_i^t - \alpha\mu_j)q_i^t + (1-z_j)\left(a_i - q_i^t - \frac{\alpha a_j}{2} + \frac{\alpha^2 q_i^t}{2}\right)q_i^t\right] + t(a_i - \alpha q_j^t - q_i^t)q_i^t$$
$$-(1-t)\left[\frac{z_j}{4}\left((a_i - \alpha\mu_j)^2 + \alpha^2\nu_j\right) + (1-z_j)\left(\frac{2a_i - \alpha a_j}{4-\alpha^2}\right)^2\right] - t\left(\frac{a_i - \alpha q_j^t}{2}\right)^2, \; i,j = 1,2, \; i \neq .$$

Where q_1^t and q_2^t are the optimal commitment quantities at time t in the tracing procedure, when the other player is also committing. These optimal actions are obtained by intersecting the following best responses:

$$q_i^t(q_j) = \frac{(1-t)\left[a_i - \frac{\alpha a_j}{2} + \alpha z_j\left(\frac{a_j}{2} - \mu_j\right)\right] + t(a_i - \alpha q_j)}{2(1-t)z_j + (1-t)(1-z_j)(2-\alpha^2) + 2t}, \; i = 1,2, \; i \neq j.$$

We do not write the explicit expressions of q_1^t and q_2^t here because they are quite enormous and do not really provide any intuition about the argument we are trying to present.

The expression $\varphi_1(t) - \varphi_2(t)$ is a non-constant function that ultimately depends on d, α and t. Furthermore, such function attains its minimum at $d = 1.02491$, $\alpha = 0.998742$, and $t = 1$. That minimum value (up to a positive constant) is 1.38778×10^{-16}, which is positive. Thus, in particular, we have that $\varphi_1(t) - \varphi_2(t)$ is positive for all $t \in [0,1]$, which means that:

$$\varphi_1(t) \geq \varphi_2(t) \; \forall t \in [0,1].$$

This proves the lemma.

■

Lemma 6.

Proof: Recall that we have defined $\phi_1(t)$ and $\phi_2(t)$ as the gains of committing versus waiting when the other player is waiting. That is,

$$\begin{aligned}
\phi_1(t) &= u_1^t(q_1^t, W_2) - u_1^t(W_1, W_2) \\
&= (1-t)u_1(q_1^t, m_2) + tu_1(q_1^t, W_2) - (1-t)u_1(W_1, m_2) - tu_1(W_1, W_2). \\
\phi_2(t) &= u_2^t(W_1, q_2^t) - u_2^t(W_1, W_2) \\
&= (1-t)u_2(m_1, q_2^t) + tu_2(W_1, q_2^t) - (1-t)u_2(m_1, W_2) - tu_2(W_1, W_2).
\end{aligned}$$

Let us analyse the behaviour of $\phi_1(t)$ and $\phi_2(t)$. In $t = 0$, we have that:

$$\begin{aligned}
\phi_1(t=0) &= u_1(q_1^0, m_2) - u_1(W_1, m_2). \\
\phi_2(t=0) &= u_2(m_1, q_2^0) - u_2(m_1, W_2).
\end{aligned}$$

And, since we are considering the case in which both players prefer to wait at the beginning, we know that $\phi_1(t=0) < 0$ and $\phi_2(t=0) < 0$. Now, we focus in $t = 1$.

$$\begin{aligned}
\phi_1(t=1) &= u_1(q_1^1, W_2) - u_1(W_1, W_2) = u_1(q_1^1, W_2) - \Pi_1^N > 0. \\
\phi_2(t=1) &= u_2(W_1, q_2^1) - u_2(W_1, W_2) = u_2(W_1, q_2^1) - \Pi_2^N > 0.
\end{aligned}$$

The last inequalities hold because $q_1^1 > q_1^N$ and $q_2^1 > q_2^N$ (clear from equations (11).

Since both $\phi_1(t)$ and $\phi_2(t)$ are continuous functions of t, we know that there must exist t_1 and t_2 such that player 1 prefers committing from t_1 onwards, and player 2 the same starting from t_2.

■

Lemma 7.

Proof: We must prove that it will be the efficient firm the first one to switch from waiting to committing. In other words, that for all $t \in [0, 1]$, we have $\phi_1(t) > \phi_2(t)$. After some algebra, it is possible to show that:

$$\phi_i(t) = (1-t)z_j\left[\left(a_i - q_i^t - \alpha\mu_2\right)q_i^t - \left(\frac{a_i - \alpha\mu_j}{2}\right)^2\right]$$
$$+ (1 - z_j + tz_j)\left[\left(\frac{2a_i - \alpha a_j - 2q_i^t + \alpha^2 q_i^t}{2}\right)q_i^t - \left(\frac{2a_i - \alpha a_j}{4-\alpha^2}\right)^2 - \frac{\alpha^2\nu_j}{4}\right],\ i,j = 1,2,\ i \neq j.$$

It can be easily seen that $\phi_1(t) - \phi_2(t)$ is a non-constant function of d, α and t. Furthermore, it attains its minimum at $d = 1.49709$, $\alpha = 0.0000640634$, and $t = 1$. The optimal value at such point (up to a positive factor) is 1.02593×10^{-9}, which is positive. In particular, this means that $\phi_1(t)$ is greater than $\phi_2(t)$ for all t in $[0, 1]$.

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