Big data, price discrimination, and collusion*

Florian Peiseler\textsuperscript{1,}† Alexander Rasch\textsuperscript{1,2} Shiva Shekhar\textsuperscript{1}

\textsuperscript{1}Düsseldorf Institute for Competition Economics (DICE), University of Düsseldorf
\textsuperscript{2}Mannheim Centre for European Economic Research (ZEW)

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Abstract

Big data allows firms to learn more accurately about various consumer characteristics. In this paper, we analyze the scope for tacit collusion in a setting in which predictive power matters for pricing: Horizontally differentiated firms can price-discriminate based on private but imperfect signals of consumers’ preferences. We find that there is a non-monotone relationship between predictive power of signals and sustainability of collusion. Starting from a low level, an increase in predictive power first facilitates collusion. However, there is a turning point from which any additional predictive power makes collusion harder to sustain. Our analysis provides important insights for competition policy. In particular, a ban on price discrimination can help to prevent collusive behavior as long as predictions are sufficiently noisy.

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\textsuperscript{†}Corresponding author. Email: \texttt{peiseler@dice.hhu.de}. Address: Düsseldorf Institute for Competition Economics (DICE), University of Düsseldorf, Universitätsstr. 1, 40225 Düsseldorf, Germany. Phone: +49 211 81 10279.
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1. Introduction

We consider a model in which firms have private but imperfect information about each consumer’s product preference before making a price offer. Consumer information is at the core of big data. The collection and processing of data enables firms to obtain an increasingly accurate picture of their consumers’ characteristics. The firms’ prediction technology then aims at using this knowledge to forecast consumers’ preferences. In our setup, we capture this process through a parameter—namely, signal precision—which can be varied exogenously and determines how accurate firms’ predictions of consumer preferences are. Firms can then use this information to price-discriminate between consumers they expect to belong to different preference segments. We analyze the impact of quality of firms’ private information, on the likelihood of (tacit) collusion.

An example for an industry in which these aspects—namely, (private) consumer data, price discrimination, and collusion—play an important role is (online) retailing. First, in order to be able to price-discriminate between different consumer segments, firms often collect data on their own customers through different channels (e.g., loyalty programs, cookies) or buy data from data-collection firms. In the US, for example, the second-largest discount store retailer Target collects consumer data from various interactions (payments by credit card, visits of web site, etc.). It uses a data-mining program to assign many different predictors to customers, one of them the so-called “pregnancy prediction”.1 The collection of consumer data and the resulting scope for price discrimination based on consumer characteristics has been hotly debated in competition-policy circles.2 Second, quality of data and precision of predictions is extremely important and crucially affects firms’ pricing decisions.3 In particular, data quality is rarely perfect. In the above-mentioned example of Target, it is reported that the “pregnancy prediction” is flawed. Pregnancy-related mailers were sent out to women for months after a miscarriage. This is due to the convergence against the mean as a result of analyzing big data. As Charles Duhigg, a journalist with the New York Times, puts it: “I can’t tell you what one shopper is going to do, but I can tell you with 90 percent accuracy what one shopper is going to do if he or she looks exactly like one million other shoppers. You

1See http://www.nytimes.com/2012/02/19/magazine/shopping-habits.html (accessed on June 2, 2017). See also Esteves (2014) for these and other examples.
3For discussions of this issue in different contexts, see Liu and Serfes (2004), Esteves (2004, chap. 2), and Colombo (2016) among others.
expect that there is some spillage there, and as a result that you will give the wrong message to a certain number of people.” Another example of misguided targeting is pointed out by the Office of Fair Trading (2010): When different users share a single computer, then firms’ offers might be received by the wrong users.

Third, collusion among in particular online retailers is an important issue. The acuteness of the problem in this multi-billion-dollar business can be seen by the recent stern warning the Competition and Market Authority (CMA) in the UK has given to online retailers against colluding among each other. The warning was issued in 2016 after the CMA found signs of price fixing among retailers in different markets on platforms such as Amazon.

In this paper, we theoretically analyze firms’ ability to sustain collusion in a model where horizontally differentiated firms can use private information to price-discriminate between different consumer segments. In our case, competing firms obtain an imperfect and independent signal on consumers’ preferences (or brand loyalty). We consider two cases—with and without price discrimination—to compare the likelihood of collusion in the two pricing regimes in terms of set inclusion.

We find that in case of price discrimination, the minimum discount factor to maintain collusive prices is non-monotone in signal precision (as well as in the level of product differentiation). More precisely, when signal precision is sufficiently low, an increase in signal quality facilitates collusion, whereas the opposite is true for a sufficiently high signal precision. For low signal precision, improving signal quality tends to have a large impact on competitive profits. As these profits decrease in signal precision, punishment becomes harsher and collusion is easier to sustain. When the level of signal quality is sufficiently high, a further increase in signal precision makes deviation relatively more profitable, as firms can charge higher prices to their loyal consumers. As a consequence, collusion becomes harder to sustain.

This finding also has consequences for the comparison of the scope for (tacit) collusion in the two pricing regimes. We find that for low levels of signal precision, collusion is more stable under a price-discrimination scheme compared to a ban on price discrimination. For large enough signal precision, collusion is harder to sustain in the discriminatory regime than under uniform pricing. This paper combines two strands of theoretical industrial organization literature: collusion among horizontally differentiated firms and third-degree (behavior-based) price discrimination with imperfect (private) information. In

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the latter strand, competitive price discrimination under best-response asymmetry leads to lower profits. Bester and Petrakis (1996) show that in a market segmented exogenously by brand loyalty, third-degree price discrimination by using coupons results in more intense competition. In a similar setup, Shaffer and Zhang (1995) show that the possibility of third-degree price discrimination by using coupons leads to a prisoner’s dilemma. Fudenberg and Tirole (2000) analyze the impact of third-degree price discrimination with perfect information. They propose a dynamic game in which learning about consumer preferences is endogenous: The first period provides information on consumer preferences, and then poaching takes place through discriminatory pricing in the second period. They also find that price discrimination results in more intense competition and hence lower profits for perfectly informed firms. While in the previous contributions, firms have perfect information on the loyalty of consumers, Esteves (2009, 2014) analyzes the impact of imperfect information on competitive profits. She shows that an increase in signal accuracy results in lower competitive profits under price discrimination. The misrecognition of consumers dampens competition, and as the loyalty signal gets more accurate, firms are more confident in their pricing strategies, which results in more competition. Colombo (2016) builds on Fudenberg and Tirole (2000) and adds noise to the second-period learning process in the form of a technology. This technology allows a firm to learn with certainty the preference of a proportion of the consumers that purchased from it. He finds that there is a non-monotone relationship between quality of information and industry profits. His article differs from Esteves (2014), because he observes the preferences of a proportion of its consumers precisely, whereas in Esteves (2014), an imperfect signal is observed. As a consequence, lower profits create incentives to commit to uniform pricing, and hence firms may prefer to collude (Stole, 2007). In our paper, we investigate on how the misrecognition effect as in Esteves (2014) affects stability of collusion.

With regard to the literature on the relationship between collusion and third-degree price discrimination, Liu and Serfes (2007) consider the impact of information precision on collusion. Information is publicly available, and an increase in information precision is defined as a higher (even) number of market segments along a linear city à la Hotelling (1929). Hence, firms are able to tell which market segment a consumer belongs to and charge a segment-specific price. The authors analyze different collusive schemes. Their main finding is that collusion becomes harder to sustain as information precision

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6Following Corts (1998), under best-response symmetries (asymmetries), firms find the same (different) groups of consumers most valuable. As a consequence, when there is price discrimination with best-response symmetries (asymmetries), static Bayesian Nash equilibrium payoffs are above (below) the level under uniform pricing.
increases. Helfrich and Herweg (2016), which is closest to our work, consider two settings with perfect information in which price discrimination leads to either best-response symmetries or best-response asymmetries. Compared to the situation in which there is a ban on price discrimination, the authors show that third-degree price discrimination helps to fight collusion under both best-response symmetries and best-response asymmetries.

The findings from the theoretical literature on the relationship between collusion and third-degree price discrimination can therefore be summarized as follows: When price discrimination is based on perfect information, theory predicts that price discrimination helps to fight (tacit) collusion. Hence, our analysis highlights that the outcomes can be fundamentally different from those in our scenario in which firms have imperfect private information.

The rest of the paper is organized as follows. Section 2 introduces the model. In Section 3, we derive the relevant payoffs for the case that firms can price-discriminate as well as for the case of a ban on price discrimination. Then, we determine the critical discount factors, which are necessary for sustaining collusion. In Section 3.3, we compare sustainability of collusion in the two pricing regimes. In Section 4, we analyze two extensions of the model and discuss the robustness of our results. Section 5 concludes.

2. Model

In this section, we first introduce the stage game, which is a static Bertrand pricing game of incomplete information. Thereafter, we describe the supergame, which is an infinite repetition of the stage game.

Stage game

We consider a model of incomplete information developed in Armstrong (2006), which is a variant of Esteves (2014, 2004, chap. 2). Consider a linear city à la Hotelling (1929) with two symmetric firms, $A$ and $B$, which are located at $\ell_A = 0$ and $\ell_B = 1$, respectively. Firms’ marginal and fixed costs are normalized to zero. They compete in prices $p_i$ with $i \in \{A, B\}$. We analyze two different pricing schemes: (i) third-degree price discrimination and (ii) uniform pricing.

Consumers of mass one are uniformly distributed along the line and derive a gross utility from buying the product, which is normalized to one. Additionally, they incur linear transport costs $\tau$ per unit of distance. Hence, when buying from firm $i$ and paying price $p_i$, a consumer located at $x$ derives net utility

$$U(x; p_i) = 1 - p_i - \tau|\ell_i - x|.$$
Consumers’ outside option is normalized to zero.\(^7\)

In our setup, there are two groups of consumers, \(L\) and \(R\), consisting of the left and right half of the linear city, respectively. Given equal prices, consumers in group \(L\) (group \(R\)) prefer firm \(A\) (firm \(B\)). Synonymously, we can call consumers in group \(L\) (group \(R\)) loyal to firm \(A\) (firm \(B\)).\(^8\)

When facing a consumer, firms do not have perfect information about the preferences, as her type is private information. Instead, each firm \(i\) receives a noisy private signal \(s_i \in \{s_L, s_R\}\) indicating the consumer’s preference. Signal precision is measured by probability \(\sigma\) and drawn independently for each firm.\(^9\) In other words, with probability \(\sigma\), information about a consumer’s preference is correctly passed on to a firm through the signal. With probability \(1 - \sigma\), the preference is misrecognized. We assume that the signal is weakly informative, that is, \(\sigma \in [1/2, 1]\). Thereby, our setup nests the following two extreme cases: (i) the signal does not convey any information, that is, \(\sigma = 1/2\), and (ii) market segments are perfectly distinguishable, that is, \(\sigma = 1\).\(^{10}\) The timing of the game is summarized below in detail.

1. Firms independently receive a private signal for any consumer along the linear city. If a consumer is located at \(x \in [0, 1/2]\), each firm receives signal \(s_L\) with probability \(\sigma\) and signal \(s_R\) with probability \(1 - \sigma\). If a consumer is located at \(x \in (1/2, 1]\), each firm receives signal \(s_R\) with probability \(\sigma\) and signal \(s_L\) with probability \(1 - \sigma\).

2. Firms simultaneously set prices. Under price discrimination, firms can condition their prices on their private signal, whereas they set a single price under uniform pricing.

3. Consumers decide from which firm to buy, and payoffs are realized.

As signals are private, firms do not know their competitor’s payoff function. Hence, we consider a game of incomplete information. In order to solve the stage game, we use the notion of Bayesian Nash equilibrium. Our tie-breaking rule is the following: Whenever a consumer values the outside option and a

\(^7\)Our results hold qualitatively if the outside option is located at each end node of the line as in Bénabou and Tirole (2016) (see the discussion below).

\(^8\)As in Esteves (2014), the definition of brand loyalty is similar to Raju et al. (1990, p. 279), where “the degree of brand loyalty is defined to be the minimum difference between the prices of the two competing brands necessary to induce the loyal consumers of one brand to switch to the competing brand”.

\(^9\)As such, the scope of information is exogenous, which is common in the literature (see Fudenberg and Tirole, 2000; Helfrich and Herweg, 2016; Liu and Serfes, 2007).

\(^{10}\)The first case hence represents the classic model, whereas the second case corresponds to a segmented market with two distinguishable segments (see Liu and Serfes, 2007 and Helfrich and Herweg, 2016).
firm equally, she chooses the firm, and in case she is indifferent between the firms, she chooses randomly.\footnote{Ties are not outcome-relevant as the distribution of consumers is atomless.}

**Dynamic game**

In order to study the scope for collusive behavior, we extend our setup to a game of infinite horizon. In the infinitely repeated game, the stage game described above is played in each period $t = 0, \ldots, \infty$. Firms are long-lived, that is, they play over the entire sequence of the infinitely repeated game. Expected payoff in period $t$ is defined as the stage game payoff plus the discounted value of the stream of future payoffs determined by the continuation game strategy profile. Firms’ common discount factor is $\delta \in (0, 1)$. Consumers are short-lived, that is, they only play for a single period and are replaced by a new cohort of consumers in the subsequent period. As a consequence, intertemporal price discrimination is not possible. Their payoff is given by their net utility in the respective period. All players are payoff-maximizing. Consumers are perfectly informed. Hence, their payoff is deterministic.

As the stage game is Bayesian, we use the notion of Bayesian Nash equilibrium when analyzing the dynamic game.\footnote{As consumers are short-lived, firms cannot learn their preferences over time. The same holds true for beliefs regarding private information of the competitor, as signals are independent across periods.}

Further, we assume that firms use grim-trigger strategies as defined in Friedman (1971) to support collusive outcomes. Thereby, we follow the related literature and can compare results. On the other hand, we want to focus on the impact of signal quality on the following trade-off for a firm: (i) long-term gains from collusive behavior compared to competitive outcomes against (ii) short-term gains by deviating unilaterally from collusive behavior. This seems plausible to us especially when thinking about tacit collusion without a certain punishment mechanism. Grim-trigger strategies generate exactly this trade-off, as punishment coincides with competitive outcomes. If, instead, optimal penal codes as in Lambson (1987) and Abreu (1988) are employed in our setup, punishment payoffs become deterministic and finite. Then, firms still trade off gains from deviation against losses from punishment, but do not take into account competitive outcomes at all by assumption.\footnote{See Appendix B for a characterization of optimal penal codes in our game.}

The stationary strategy profile can be summarized as follows:

- In the starting period $t = 0$, each firm charges the collusive price. In any subsequent period $t = 1, \ldots, \infty$, each firm
charges the collusive price as long as it does not observe any other price in period $t-1$ and
plays Bayesian Nash equilibrium strategies else.

- Consumers buy from the firm providing the highest net utility if it weakly exceeds the value of the outside option. If a consumer is indifferent between the two firms, she chooses randomly.

In order to verify whether the suggested strategy profile constitutes a perfect Bayesian equilibrium, we need to verify that the one-shot-deviation principle (OSDP) is satisfied (for a formal argument, see Hendon et al., 1996). Given firms’ strategies and beliefs over consumers’ preferences and the respective competitors private information, this is true if and only if the following inequality is satisfied in any period $t$:

$$\frac{\pi^c}{1-\delta} \geq \frac{\pi^d + \delta \pi^*}{1-\delta},$$

where $\pi^*$, $\pi^c$, and $\pi^d$ denote competitive (punishment) profits, collusive profits, and deviation profits, respectively. From this it follows that the critical discount factor is defined by

$$\delta \geq \frac{\pi^d - \pi^c}{\pi^d - \pi^*} =: \bar{\delta}.$$  \hspace{1cm} (2)

All things equal, a lower (higher) punishment or deviation profit facilitates collusion (makes collusion harder to sustain), that is, the set of discount factors which satisfy OSDP becomes larger (smaller). The opposite is true for the respective change in the collusive profit. Put differently, lower (higher) gains from defecting (i.e., $\pi^d - \pi^c$) and higher (lower) losses from punishment (i.e., $\pi^c - \pi^*$) make collusion easier (harder) to sustain.

Throughout the analysis, we focus on equilibria in which the market is covered, that is, all consumers along the line buy from one of the two firms. For this purpose, we impose the following assumption on consumers’ transport costs:

**Assumption 1.** $\tau \in [0, 2/3]$.

Assumption 1, which is common in the related literature, guarantees that the market is fully served when price discrimination is banned.\textsuperscript{14,15}

\textsuperscript{14}For the case of price discrimination, the market is covered for larger values of the transport costs as prices tend to be lower. To ensure better comparability, we use the more restrictive upper bound on the transport-cost parameter.

\textsuperscript{15}Instead, one could follow Bénabou and Tirole (2016) by assuming that the outside option
3. Analysis and results

In this section, we derive the critical discount factors for the case of price discrimination and the case in which price discrimination is banned, that is, uniform pricing.

3.1. Price discrimination

In order to evaluate firms’ ability to sustain collusion under price discrimination, we need to derive the profits under competition, deviation, and collusion. Firms want to condition their prices on the signal they receive as long as it is informative: After observing signal $s_L$, firm $i$ charges $p_{i,L}$, and after observing $s_R$, it charges $p_{i,R}$. For demand to be well-defined, suppose for now that given prices of firm $B$, it has to hold for firm $A$ that $p_{B,L} \leq p_{A,R} \leq p_{A,L} \leq p_{B,R}$ and given prices of firm $A$, it has to hold for firm $B$ that $p_{A,R} \leq p_{B,L} \leq p_{B,R} \leq p_{A,L}$. The intuition behind the restrictions is that, on the one hand, a firm does not find it profitable to charge lower prices from its loyal consumers than its rival. Neither it finds it profitable to charge lower prices from consumers that prefer the firm than from consumers that prefer its competitor. On the other hand, a firm cannot attract any consumer that is loyal to its competitor by charging a higher price. The remaining conditions are without loss of generality and can be specified differently. It will be shown later in this subsection that equilibrium prices indeed satisfy all conditions.

In order to derive expected demand of a firm conditional on its private signal, we need to distinguish all possible outcomes, where an outcome is characterized by a tuple $(s_j, s_k)$ with $j, k \in \{L, R\}$, and where the first (second) element is the signal of firm $A$ (firm $B$). Since signals are independently drawn for each consumer, firms can either receive identical signals ($j = k$) or different signals ($j \neq k$), that is, the set of possible signal realizations is given by $S := \{(s_L, s_L), (s_L, s_R), (s_R, s_L), (s_R, s_R)\}$. As firms condition their prices on their private signal, they have to take into account four different indifferent consumers, which determine the probability of winning a certain consumer or, equivalently, the market share in a segment, for any signal tuple. To this end, $\tilde{x}_1$ denotes the indifferent consumer for tuple $(s_L, s_L)$, $\tilde{x}_2$ for $(s_R, s_L)$, $\tilde{x}_3$ for $(s_L, s_R)$, and $\tilde{x}_4$ for $(s_R, s_R)$. Solving for each, we get

$$1 - p_{A,L} - \tau \tilde{x}_1 = 1 - p_{B,L} - \tau (1 - \tilde{x}_1) \iff \tilde{x}_1 = \frac{1}{2} - \frac{p_{A,L} - p_{B,L}}{2\tau},$$

is costly, that is, it is located at either end of the linear city. Then, Assumption 1 would not be needed. However, this would not change our results qualitatively but make the comparison to the above mentioned literature less clean.
\[ 1 - p_{A,L} - \tau \tilde{x}_2 = 1 - p_{B,R} - \tau (1 - \tilde{x}_2) \iff \tilde{x}_2 = \frac{1}{2} - \frac{p_{A,L} - p_{B,R}}{2\tau}, \]
\[ 1 - p_{A,R} - \tau \tilde{x}_3 = 1 - p_{B,L} - \tau (1 - \tilde{x}_3) \iff \tilde{x}_3 = \frac{1}{2} - \frac{p_{A,R} - p_{B,L}}{2\tau}, \]
and
\[ 1 - p_{A,R} - \tau \tilde{x}_4 = 1 - p_{B,R} - \tau (1 - \tilde{x}_4) \iff \tilde{x}_4 = \frac{1}{2} - \frac{p_{A,R} - p_{B,R}}{2\tau}. \]

Due to the restriction of the set of feasible prices above, it holds true that \( \tilde{x}_1, \tilde{x}_3 \in [0, 1/2] \) and \( \tilde{x}_2, \tilde{x}_4 \in [1/2, 1] \). For firm A, the probability of winning consumer \( x \in L \) given \((s_L, s_L)\) is equal to 2\(x_1\). In the same firm and segment, the probability of winning the consumer given \((s_L, s_R)\) is equal to 2\((x_2 - 1/2)\).

In both cases, the winning probability is equivalent to the firm’s expected market share in segment \( L \). The remaining cases can be derived analogously.

The notion of Bayesian Nash equilibrium requires that firm \( i \)—after receiving a signal—updates its beliefs regarding the respective consumer’s actual preference and regarding the signal of its competitor. As signal realizations are independent across firms and periods, the updating process is independent in each stage game. A firm’s posterior belief that a consumer prefers firm A given signal \( s_L \) is

\[ \Pr(L | s_L) = \frac{\Pr(s_L | L) \Pr(L)}{\Pr(s_L | L) \Pr(L) + \Pr(s_L | R) \Pr(R)} = \sigma, \]

which is equal to the precision of the signal due to symmetry. Conditional on this, firm \( i \)’s posterior belief that firm \( j \) has received signal \( s_L \) is equal to the conditional probability of this event, namely \( \sigma \). In the remaining cases, beliefs are updated analogously.

Then, firm A’s expected demand conditional on receiving signal \( s_L \) can be derived as

\[ D_A(p_{A,L}, p_{B,L}, p_{B,R} | s_L) = \sigma (2\sigma \tilde{x}_1 + 1 - \sigma) + 2\sigma (1 - \sigma) \left( \tilde{x}_2 - \frac{1}{2} \right) \]
\[ = \sigma \left( 1 - \frac{p_{A,L} - \sigma p_{B,L} - (1 - \sigma)p_{B,R}}{\tau} \right). \quad (3) \]

Similarly, conditional on receiving signal signal \( s_R \), firm A’s expected demand can be derived as

\[ D_A(p_{A,R}, p_{B,L}, p_{B,R} | s_R) = (1 - \sigma) (2\sigma \tilde{x}_3 + 1 - \sigma) + 2\sigma^2 \left( \tilde{x}_4 - \frac{1}{2} \right) \]
\[ = 1 - \sigma - \frac{\sigma p_{A,R} - \sigma (1 - \sigma)p_{B,L} - \sigma^2 p_{B,R}}{\tau}. \quad (4) \]
Expected demand for firm $B$ conditional on its signal realization can be derived analogously.

**Competition**

We start by analyzing the competitive profits, that is, the static Bayesian Nash equilibrium payoff of the stage game as defined in Section 2. These are used as punishment payoffs in the dynamic game.\(^{16}\) The maximization problem of firm $i$ is given as

$$
\max_{p_i,L,p_i,R} \mathbb{E} [\pi_i] = p_{i,L} \Pr(s_i = s_L) D_{i,L}(p_{A,L},p_{B,L},p_{B,R}|s_L) + p_{i,R} \Pr(s_i = s_R) D_{i,R}(p_{A,R},p_{B,L},p_{B,R}|s_R),
$$

where $\Pr(s_i = s_L) = \Pr(s_i = s_R) = 1/2$. Differentiating with respect to prices and solving the system of first-order conditions gives symmetric equilibrium prices of

$$
p^{*}_{A,L} = p^{*}_{B,R} = \frac{2\tau}{1 + 2\sigma} \quad \text{and} \quad p^{*}_{A,R} = p^{*}_{B,L} = \frac{\tau}{\sigma(1 + 2\sigma)},
$$

where $p^{*}_{A,R} < p^{*}_{A,L}$ and $p^{*}_{B,L} < p^{*}_{B,R}$ hold as long as the signal is informative. Then, price discrimination allows firms to set higher prices for those consumers who are signaled to be located more closely to their own location, that is, consumers with a higher willingness to pay for their product. Thereby, the market is segmented into four as in Fudenberg and Tirole (2000) under informative signals, although firms can only distinguish between two consumer groups. In each half, there are consumers served by their preferred firm, and consumers poached by the less preferred firm, i.e., $0 < \tilde{x}_1^* < \tilde{x}_2^* = \tilde{x}_3^* = 1/2 < \tilde{x}_4^* < 1$ $\forall \sigma \in (1/2,1]$.

The equilibrium profit for each firm amounts to

$$
\pi^* = \frac{\tau (1 + 4\sigma^2)}{2\sigma (1 + 2\sigma)^2}.
$$

We observe that firm profits are decreasing in the signal precision. As Esteves (2014) points out, an increase in signal precision has two opposing effects: On the one hand, misrecognition of consumers decreases, which means that a firm can charge more from its loyal consumers, while reducing the price to those consumers who are loyal to its rival. In other words, a firm can poach more effectively. On the other hand, since the rival behaves more aggressively as well when poaching loyal consumers, a firm optimally responds by reducing its

\(^{16}\)The results from this part are equivalent to Armstrong (2006).
prices. In this setup, it turns out that the latter effect (competition) outweighs the increase in prices due to reduction in misrecognition (information). Hence, competition is intensified with a rise in signal precision. As a result, for any \( \sigma \in (1/2, 1] \), profits are strictly lower than static Bayesian Nash equilibrium profits under uniform pricing, as we have best-response asymmetries.

Collusion

Under full collusion, firms maximize industry profits by minimizing total transport costs, that is, firms divide the market in two and each firm serves its own turf. In our game, this allocation can only be induced by charging symmetric prices. As firms try to extract the maximal surplus from consumers net of transport costs, it is not optimal to attract consumers in the competitor’s turf. Put differently, firms will not price-discriminate based on private information about consumers’ preferences. Instead, they will set a single price for all consumers such that the marginal consumer located at \( 1/2 \) is indifferent between buying and not buying, that is, \( 1 - p^c - \tau |\ell - 1/2| = 0 \). We summarize these considerations in the following lemma:

**Lemma 1.** Collusive prices and profits are given by

\[
p^c = 1 - \frac{\tau}{2}
\]

and

\[
\pi^c = \frac{1}{2} - \frac{\tau}{4}.
\]

We observe that price discrimination cannot lead to higher profits compared to uniform pricing, as firms can only distinguish two consumer groups.\(^{17}\)

Deviation

In order to characterize the optimal deviation strategy, we need to define the following thresholds for \( \tau \):\(^{18}\)

\[
\tau_1 := \frac{2(1 - \sigma)}{5 - 3\sigma}, \quad \tau_2 := \frac{2\sigma}{2 + 3\sigma}, \quad \text{and} \quad \tau_3 := \frac{2(1 - \sigma)}{1 + \sigma}.
\]

\(^{17}\)In this setup, firms do not price-discriminate under collusion, which is also the case in Helfrich and Herweg (2016) and Liu and Serfes (2007) (with two segments). This is due to the fact that we only allow for a left and a right market, i.e., two signals. The present model could easily be extended to more signals, which would yield price discrimination also under collusion. At the same time, results would not change qualitatively (in particular, see the deviation incentives for low values of signal precision and transport costs below). For tractability reasons, we restrict our attention to two signals.

\(^{18}\)The derivation of these thresholds is part of the proof of Lemma 2 in Appendix A.
It is easily checked that \( \tau_1, \tau_2, \tau_3 \in [0, 2/3] \) for any \( \sigma \in [1/2, 1] \) and that \( \tau_2 \leq \tau_3 \) for \( \sigma \leq 1/\sqrt{2} \). The following lemma characterizes optimal deviation behavior:

**Lemma 2.** The optimal deviation from collusive prices yields the following prices and profits, which are continuous and differentiable in both \( \sigma \) and \( \tau \):

\[
p_{A,L}^d = p_{B,R}^d = \begin{cases} 
1 - \frac{3\tau}{2} & \text{if } \tau \in [0, \tau_1] \\
\frac{1}{2} + \frac{\tau(3\sigma - 1)}{4(1 - \sigma)} & \text{if } \tau \in (\tau_1, \tau_3], \\
1 - \frac{\tau}{2} & \text{if } \tau \in (\tau_3, 2/3]. 
\end{cases}
\]

\[
p_{A,R}^d = p_{B,L}^d = \begin{cases} 
1 - \frac{3\tau}{2} & \text{if } 0 \leq \tau \leq \tau_2, \\
\frac{1}{2} - \frac{\tau(3\sigma - 2)}{4\sigma} & \text{if } \tau_2 < \tau \leq 2/3, 
\end{cases}
\]

and,

\[
\pi^d = \begin{cases} 
1 - \frac{3\tau}{2} & \text{if } \tau \in [0, \tau_1], \\
\frac{3\tau(1 - \sigma) + 2(1 - \sigma)^2 - 32\tau^2}{32\tau(1 - \sigma)} & \text{if } \tau \in (\tau_1, \min\{\tau_2, \tau_3\}], \\
\frac{\tau}{8(1 - \sigma)} + \frac{4(\sigma + 1) - 15\tau^2}{32\tau} & \text{if } \sigma \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right] \land \tau \in (\tau_2, \tau_3], \\
\frac{\tau^2(2 - \sigma)^2 + 4\sigma^2(\sigma + 1) + 8\sigma(1 - \sigma)}{32\tau^2} & \text{if } \sigma \in \left(\frac{1}{\sqrt{2}}, 1\right] \land \tau \in (\tau_3, \tau_2], \\
\frac{\tau^2(2 - \sigma)^2 + 4\sigma^2(\sigma + 1) + 8\sigma(1 - \sigma)}{32\tau^2} & \text{if } \tau \in \left(\max\{\tau_2, \tau_3\}, \frac{2}{3}\right]. 
\end{cases}
\]

**Proof.** See Appendix A.

Figure 1 divides all combinations of parameter values of \( \sigma \) and \( \tau \) into five regions corresponding to the cases in Lemma 2. Generally speaking, the segments result from a deviating firm’s decision whether to serve all consumers or only a portion for a given signal, which is a function of both signal quality and transport costs.

When transport costs are (very) small (i.e., \( \tau \in [0, \tau_1] \)), a deviating firm finds it profitable to set a uniform price for both consumer groups to cover the whole market. When transport costs are of low or intermediate value, and signal quality is not too precise (i.e., \( \tau \in (\tau_1, \min\{\tau_2, \tau_3\}] \)), a deviating firm sets a relatively low price for those consumers who were signaled to be farther away from its own location and serves all of them. Trying to serve all consumers with a signal to be nearer to a deviating firm’s location would be too costly—it forgoes some demand and sets a higher price instead.

When transport costs are of intermediate value, and signal precision is good (i.e., \( \tau \in (\tau_3, \tau_2] \)), a deviating firm can target near consumers by matching the competitor’s collusive price. Those consumers with a signal to be located more closely to the competitor again pay a relatively low price, such that this market segment is fully served by the deviating firm.
Figure 1: Characterization of deviation strategies.

Note: The dotted horizontal line at 2/7 gives the threshold below which a deviating firm wants to serve the whole market in the case of uniform pricing.

For high transport costs and very low to intermediate signal precision (i.e., \( \tau \in (\tau_2, \tau_3] \)), it is too costly for a deviating firm to serve all consumers whose signal implies a location closer to the competitor. At the same time, because signals are too imprecise, a deviating firm does not match the collusive price for those consumers who are signaled to be near to its own location but sets a lower price. Now, when transport costs are even higher and/or signal quality is better or even perfect (i.e., \( \tau \in (\max\{\tau_2, \tau_3\}, 2/3] \)), a deviating firm’s pricing strategy changes compared to the previous case in that it sets the collusive price for those consumers whose signal indicated a location close to its own.

Critical discount factor

Using the profits derived in the three above scenarios, we can determine the critical discount factor \( \bar{\delta} := \bar{\delta}(\sigma, \tau) \) as defined in Condition (2). The following proposition characterizes the critical discount factor:

**Proposition 1.** When firms can price-discriminate, the critical discount factor \( \bar{\delta} \) is a continuous and differentiable function of \( \sigma \) and \( \tau \) with the following properties:
(i) $\bar{\delta}$ is non-monotone in the signal quality such that $\partial \bar{\delta} / \partial \sigma < 0 \ (> 0)$ holds for low (high) values of $\sigma$.

(ii) $\bar{\delta}$ is non-monotone in the transport costs such that $\partial \bar{\delta} / \partial \tau < 0 \ (> 0)$ holds for low (high) values of $\tau$.

Proof. See Appendix A.

Let us have a closer look at the intuition behind these findings. We know from the analysis above that the collusive profit is independent of signal quality, whereas the deviation profit and the Bayesian Nash equilibrium profit depend on it. More precisely, for a given value of the transport-cost parameter, the deviation profits are weakly increasing in signal quality, as targeting consumers becomes easier. At the same time, Bayesian Nash equilibrium profits are falling in signal quality, as competition gets fiercer. Hence, increasing signal quality has opposing effects on the critical discount factor, as both the gains from deviation and the losses from punishment increase. We know from the benchmark results in Liu and Serfes (2007) and Helfrich and Herweg (2016) that for perfect signal precision, collusion is harder to sustain under price discrimination than under uniform pricing, that is, $\bar{\delta}(1/2, \tau) < \bar{\delta}(1, \tau)$.

Now consider the case in which signal quality improves starting from a low level of signal precision. In this case, the gain from defecting increases relatively slower than the loss from punishment. For the case with relatively low transport costs, this is intuitive. A deviating firm finds it profitable to serve the whole market irrespective of the signal precision, meaning that the deviation profit and thus the gain from cheating does not change. This is due to the fact that the optimal deviating price is determined by the consumer located at the other firm’s location and is hence independent of the signal quality. In contrast, competition is intensified, that is, competitive profits decrease, and the loss from punishment increases. As a result, collusion is facilitated.

The case in which transport costs are relatively high is more involved. In Liu and Serfes (2007), where signals are perfect and the number of distinguishable market segments determines the quality of information, improved information makes deviation relatively more attractive. Whereas there is only one stabilizing effect strengthening competition in the punishment case, there are two destabilizing effects which render deviation very profitable. First, a deviating firm can target consumers more accurately. Second, undercutting is cheaper as collusive prices are higher. In our case, however, only the first destabilizing effect is present under deviation, because collusive prices are not affected by signal quality. Moreover, this remaining destabilizing effect is less pronounced for low levels of the signal quality. As a deviating firm often misrecognizes consumers due to poor signal quality, it must set lower prices for both consumer
groups, resulting in lower profits from deviation.

As signal quality becomes more accurate, deviation becomes relatively more profitable, as a deviator can optimistically charge the collusive price to those consumers who are signaled to be near to its own location. As a consequence, collusion becomes harder to sustain.

With regard to the impact of the transport costs, we point out that, compared to the case with uninformative signals and hence no price discrimination, there is an important difference. When transport costs are very high, deviation is very difficult and requires a large price cut to gain market share, which drastically reduces profits from inframarginal loyal consumers. As pointed out before, when a deviating firm has at least some reliable information on its loyal consumers, it can target those with the same price as under collusion and offer a much lower price only to those consumers who are loyal to its competitor. This makes deviation relatively more profitable. At the same time, there are no such drastic effects under competition and collusion. As a result, collusion becomes harder to maintain.

Proposition 1 provides new insights for competition policy. Under competition an increase in signal precision leads to lower consumer prices due to best-response asymmetries. Also, when signals are perfect, both prices and the likelihood of collusive behavior are lowest. Both effects benefit consumers.

At the same time, however, we know from the discussion above, that, at a relatively low level, a marginal increase of signal quality can make collusive behavior more likely. In other words, any policy in the context of big data that further deregulates access to or usage of consumer data\footnote{To a certain extent, it seems natural to assume a positive relation between the amount and variety of available data and predictive power. Yoganarasimhan (2017) provides evidence for this relation in the context of search queries. She finds that personalized search, especially long-term and across-session, helps to improve accuracy of suggested results significantly.} so that firms gain predictive power can lead to collusive behavior instead of more competition. Hence, if the industry is asking for such deregulations, regulators and policy makers should be alerted. This is due to the fact that under competition better information for a single firm only is always beneficial. In contrast, better information for the whole industry drives down firms’ competitive profits, but firms may be able to coordinate on collusive prices under the new policy, whereas this is not possible without the change. For relatively high levels, a marginal increase of signal quality can make collusive behavior less likely. Then, any policy that is concerned with consumer privacy and is restrictive in a sense that firms lose predictive power can come at the cost of collusive behavior.
3.2. Ban on price discrimination

The above case nests the scenario in which firms are not allowed to price-discriminate, because outcomes are the same as in the situation in which signals are uninformative (i.e., \( \sigma = 1/2 \)). Hence, punishment payoffs reduce to

\[ \pi_{ban}^* = \frac{\tau}{2}, \]

collusive payoffs to

\[ \pi_{ban}^C = \pi^C = \frac{1}{2} - \frac{\tau}{4}, \]

and deviation payoffs to

\[ \pi_{ban}^D = \begin{cases} 
1 - \frac{3\tau}{2} & \text{if } 0 \leq \tau \leq \frac{\tau_1}{2}, \\
\frac{1}{8} + \frac{\tau}{32} + \frac{1}{8\tau} & \text{if } \frac{\tau_1}{2} < \tau \leq \frac{2 \tau_1}{3}.
\end{cases} \]

Given these profits and Condition (2), it immediately follows that the critical discount factor is given as

\[ \bar{\delta}_{ban} = \begin{cases} 
\frac{2 - 3\tau}{4(1 - 2\tau)} & \text{if } 0 \leq \tau \leq \frac{\tau_1}{2}, \\
\frac{2 - 3\tau}{2 + 5\tau} & \text{if } \frac{\tau_1}{2} < \tau \leq \frac{2 \tau_1}{3}.
\end{cases} \]

By construction, \( \bar{\delta}_{ban} \) is independent of \( \sigma \). It decreases in the transport-cost parameter, that is, \( \partial \bar{\delta}_{ban} / \partial \tau < 0 \), as established in Chang (1991).

3.3. Comparison

We can now compare the critical discount factors in the two scenarios with and without price discrimination. As the critical discount factors depend on the profits in the three scenarios, we start with these. Profits are to a large extent affected differently by the possibility to price-discriminate. As pointed out above, the collusive profits are the same in the two scenarios. As far as the punishment profits are concerned, it holds that punishment is harsher under (meaningful) price discrimination, that is, \( \pi_{ban}^* < \pi_{ban}^C \) for \( \sigma \in (1/2, 1] \). Hence, collusion tends to be facilitated.

At the same time, it is true that deviation profits are the same with and without price discrimination as long as the transport costs are sufficiently small, that is, \( \tau \in [0, \tau_1] \). As pointed out before, this is due to the fact that in this case, a deviating firm finds it optimal to cover the whole market in both scenarios. When transport costs are greater than this threshold, it holds that a deviating firm makes a higher profit under price discrimination, that is, \( \pi_{ban}^D < \pi^D \) for \( \tau \in (\tau_1, 2/3] \). As a deviating firm has better information
regarding its consumers’ locations when price discrimination is possible, it can better fine-tune its deviating strategy, which yields higher profits. Hence, collusion tends to be harder to sustain.

As a result—and in line with the literature on the impact of price discrimination on collusion—there are opposing effects. The following proposition answers the question which of these effects dominates.

**Proposition 2.** For any \( \tau \in (0, 2/3) \), there exists signal quality \( \tilde{\sigma}(\tau) \in (1/2, 1) \) such that for \( \sigma = \tilde{\sigma}(\tau) \), we have \( \delta = \delta_{\text{ban}} \). Furthermore, for any \( \sigma \leq \tilde{\sigma}(\tau) \), it holds true that \( \delta \leq \delta_{\text{ban}} \).

**Proof.** See Appendix A.

Figure 2 illustrates the finding for all permissible parameter values of transport costs and signal quality. When signal quality is so poor that signals do not provide any information (i.e., \( \sigma = 1/2 \)), price discrimination is not feasible. Hence, the critical discount factors are the same. Recall again that we know from Liu and Serfes (2007) and Helfrich and Herweg (2016) that given perfect signal quality (i.e., \( \sigma = 1 \)), which divides the linear city into two clearly distinguishable markets, collusion is sustainable for a smaller range of the discount factor under price discrimination than under no price discrimination. Hence, allowing price discrimination helps to fight collusion.

Moreover, as highlighted in Proposition 1, starting from a low value of signal precision, an increase in signal quality results in a lower critical discount factor. As a consequence, banning price discrimination helps to fight collusive behavior in this case, as the critical discount factor under no price discrimination remains unchanged. For (very) low transport costs (i.e., \( \tau \in [0, \tau_1] \)), this insight can also be immediately traced back to the difference in competitive profits. In this case, the two scenarios with and without price discrimination only differ in the punishment phase. As competitive profits are lower under price discrimination, collusion is facilitated.

The impact of signal accuracy on the incentives to collude in the two pricing scenarios is illustrated in Figure 3. Under uniform pricing, none of the profits changes in signal quality. Hence, the critical discount factor is also independent of it, as depicted by the straight dashed lines in both panels. In contrast, as pointed out before, a change in signal quality affects the critical discount factor under price discrimination. For low levels of signal accuracy, the reduction in punishment profits outweighs the rise in deviation profits (if any). As a result, a rise in signal accuracy leads to higher collusion incentives. For high enough levels of signal accuracy, the rise in deviation profits looms larger than the fall in punishment profits when signal accuracy rises. Hence, collusion is harder to sustain.
Figure 2: Comparison of the critical discount factors with and without price discrimination for all permissible parameter values.

Note: For those parameter combinations represented by the solid lines, the two critical discount factors coincide, that is, $\bar{\delta} = \bar{\delta}_{\text{ban}}$. The dotted lines separate the different regions with respect to the deviation strategies for the cases with and without price discrimination.

From the analysis, we know that collusion without price discrimination is harder to sustain for low levels of transport costs. We know that in this case, collusive and deviation profits are equal under both pricing regimes. As punishment is harsher when firms can price-discriminate, this results in a lower critical discount factor. However, we also know from previous contributions that collusion is harder to sustain under price discrimination for perfect signals. Therefore, at some value for the transport costs, collusion is easier to sustain without price discrimination.

4. Robustness

In this section, we test the robustness of our main results by relaxing some of the assumptions imposed on the signal structure. More precisely, we consider the cases of asymmetric signal quality and correlated signals.
4.1. Asymmetric signal quality

In this subsection, we relax the assumption of symmetric information accuracy. Similar to Esteves (2014), we assume that the signal a firm receives is a function of the respective consumer’s preference. We consider the following case: The signal a firm receives when facing a loyal consumer is weakly more precise than the signal it receives when facing a disloyal consumer. Let us denote the probability that the signal is correct if the consumer is loyal (disloyal) by $\sigma_1$ ($\sigma_2$) and assume that $1/2 \leq \sigma_2 \leq \sigma_1$. Thereby, we address the concern that a firm might know most about the characteristics of its loyal consumers and hence should be able to identify these with higher probability, which can also be interpreted as a short-cut approach to modeling consumers who live for more than a single period and firms which have access to an imperfect tracking technology similar to the one defined in Colombo (2016).

Consider the set $S$, which contains all possible signal tuples $(s_j, s_k)$, and let $f(s_j, s_k | x \in l)$ denote the joint probability density function conditional on consumer $x$’s preference $l \in \{L, R\}$. We impose the following assumption on the functional form of $f(\cdot)$:
Assumption 2.

\[ f(s_j, s_k | x \in L) = \begin{cases} 
\sigma_1 \sigma_2 & \text{for } (s_L, s_L), \\
\sigma_1 (1 - \sigma_2) & \text{for } (s_L, s_R), \\
(1 - \sigma_1) \sigma_2 & \text{for } (s_R, s_L), \\
(1 - \sigma_1) (1 - \sigma_2) & \text{for } (s_R, s_R),
\end{cases} \]

and

\[ f(s_j, s_k | x \in R) = \begin{cases} 
(1 - \sigma_2) (1 - \sigma_1) & \text{for } (s_L, s_L), \\
(1 - \sigma_2) \sigma_1 & \text{for } (s_L, s_R), \\
\sigma_2 (1 - \sigma_1) & \text{for } (s_R, s_L), \\
\sigma_2 \sigma_1 & \text{for } (s_R, s_R). 
\end{cases} \]

The density function under Assumption 2 is well-defined and nests the extreme case of symmetric signals (for \( \sigma_1 = \sigma_2 = \sigma \)). As before, after observing signal \( s_i \), firm \( i \) has to infer on the consumer’s actual preference and on the signal \( s_j \) received by its competitor. Suppose firm \( i \) receives signal \( s_L \). Applying Bayes’ rule, its updated belief that a consumer prefers firm \( A \), and its competitor has received the same signal is

\[
\Pr(s_L, L | s_L) = \frac{f(s_L, s_L | L) \Pr(L)}{f_{s_i}(s_L | L) \Pr(L) + f_{s_i}(s_L | R) \Pr(R)} = \frac{\sigma_1 \sigma_2}{1 + \sigma_1 - \sigma_2},
\]

where \( f_{s_i} \) denotes the marginal distribution of \( s_i \). In the remaining cases, beliefs are updated analogously. Given beliefs, we can specify each firm’s maximization problem and determine mutual best responses similarly to the main analysis (see the Appendix). Firms optimally set prices equal to

\[
p_{A,L}^* = p_{B,R}^* = \frac{2 \tau \sigma_1}{\sigma_2 + 2 \sigma_1 \sigma_2} \quad \text{and} \quad p_{A,R}^* = p_{B,L}^* = \frac{\tau}{\sigma_2 + 2 \sigma_1 \sigma_2},
\]

where \( p_{A,R}^* < p_{A,L}^* \) and \( p_{B,L}^* < p_{B,R}^* \) as long as the signal is informative. The resulting equilibrium payoff for each firm amounts to

\[
\pi^* = \frac{\tau (1 + 4 \sigma_1^2)}{2 \sigma_2 (1 + 2 \sigma_1)^2}.
\]

These profits serve as punishment payoffs in the dynamic game as defined in Section 2 and equal those derived in Section 3 for \( \sigma_1 = \sigma_2 = \sigma \) by construction. The intuition from the symmetric case can be misleading here by suggesting a similar relation between punishment payoffs and average signal quality. In fact, we observe that the more asymmetric the signal quality is, the higher the
punishment payoffs are—namely, they rise in $\sigma_1$ and fall in $\sigma_2$. When $\sigma_1$ increases, firms can better identify their loyal consumers allowing for an increase of their price. On the other hand, when $\sigma_2$ decreases, firms more often misrecognize their disloyal consumers leading to less aggressive poaching, as costly mistakes become more likely. Overall, signal asymmetry softens competition. As deviation profits are affected in the same way (see the proof of Proposition 3), it is not clear from an ex-ante perspective how signal asymmetry translates into the critical discount factor $\bar{\delta}_{asy}$. The following proposition summarizes our result:

**Proposition 3.** For any $\sigma_2 < \sigma_1$, the critical discount factor $\bar{\delta}_{asy}$ is strictly larger compared to both cases of symmetric signal quality $\sigma = \sigma_1$ and $\sigma = \sigma_2$. In addition, $\bar{\delta}_{asy}$ is non-monotone in $\sigma_1$ and $\sigma_2$.

**Proof.** See Appendix A.

By construction, the critical discount factors in the symmetric and asymmetric case are equivalent for $\sigma_1 = \sigma_2 = \sigma \in [1/2, 1]$. Starting from $\sigma_1 = \sigma_2 = 1/2$, we can see from the proof of Proposition 3 that a marginal increase in both dimensions leads to a marginal reduction of $\bar{\delta}_{asy}$. From continuity and Proposition 2, it immediately follows that we can always find $1/2 \leq \sigma_2 < \sigma_1$, such that $\bar{\delta}_{asy} < \bar{\delta}$. Then, collusion is more likely in terms of set inclusion if price discrimination is permitted compared to the case of no price discrimination. The corollary below summarizes this argument:

**Corollary 1.** For $\sigma_2 < \sigma_1$, a ban on price discrimination helps to fight collusion if signals are sufficiently noisy.

### 4.2. Correlated signals

In this subsection, we relax the assumption of independent signal realizations by allowing for positive correlation of the private signals received by the firms. This is natural, as firms might, for instance, use similar algorithms in order to infer on consumer types from available data or obtain consumer data from similar sources.

Consider the set of all signal tuples $S$ and let $g(s_j, s_k|x \in l)$ denote the joint probability density function conditional on consumer $x$’s preference $l \in \{L, R\}$. We assume the following functional form of $g(\cdot)$:
Assumption 3.

\[
g(s_j, s_k | x \in L) = \begin{cases} 
\sigma^2 + \gamma & \text{for } (s_L, s_L), \\
\sigma(1 - \sigma) - \gamma & \text{for } (s_L, s_R), (s_R, s_L), \\
(1 - \sigma)^2 + \gamma & \text{for } (s_R, s_R), 
\end{cases}
\]

and

\[
g(s_j, s_k | x \in R) = \begin{cases} 
(1 - \sigma)^2 + \gamma & \text{for } (s_L, s_L), \\
\sigma(1 - \sigma) - \gamma & \text{for } (s_L, s_R), (s_R, s_L), \\
\sigma^2 + \gamma & \text{for } (s_R, s_R), 
\end{cases}
\]

where \( \gamma \in [0, \sigma (1 - \sigma)] \) measures the degree of correlation.

The density function under Assumption 3 is well-defined and nests the two extreme cases: (i) independent signals (for \( \gamma = 0 \)) and (ii) perfectly correlated signals (for \( \gamma = \sigma (1 - \sigma) \)). The second case is equivalent to a model with imperfect public information about consumer preferences. It is easily checked that the interval \([0, \sigma (1 - \sigma)]\) is non-empty for \( \sigma \in [1/2, 1) \). In the following, we solve for the Bayesian Nash equilibrium of the stage game. As before, after observing signal \( s_i \), firm \( i \) has to infer on the consumer’s actual preference and on the signal of its competitor. For illustration, suppose that firm \( i \) receives signal \( s_L \). Applying Bayes’ rule, its posterior belief that a consumer prefers firm A, and firm \( j \) receives the same signal is

\[
\Pr(s_L, L | s_L) = \frac{g(s_L, s_L | L) \Pr(L)}{g_s(s_L | L) \Pr(L) + g_s(s_L | R) \Pr(R)} = \sigma^2 + \gamma,
\]

where \( g_s \) denotes the marginal distribution of \( s_i \). In the remaining cases, beliefs are updated similarly. Given beliefs, we can specify each firm’s maximization problem and determine mutual best responses analogously to the main analysis (see the Appendix). Firms optimally set prices equal to

\[
p_{A,L}^* = p_{B,R}^* = \frac{\tau (\gamma + 2\sigma^2)}{\sigma (2\gamma + \sigma + 2\sigma^2)} \quad \text{and} \quad p_{A,R}^* = p_{B,L}^* = \frac{\tau (\gamma + \sigma)}{\sigma (2\gamma + \sigma + 2\sigma^2)},
\]

where \( p_{A,L}^* < p_{A,R}^* \) when the signal is informative. Resulting equilibrium profits for each firm are

\[
\pi^* = \frac{\tau (2\gamma^2 + 2\gamma \sigma (2\sigma + 1) + 4\sigma^4 + \sigma^2)}{4\sigma (2\gamma + 2\sigma^2 + \sigma^2)^2}.
\]

These profits are the punishment payoffs in the dynamic game as defined in Section 2. By construction, punishment payoffs are equal to those derived
in Section 3 for $\gamma = 0$. Furthermore, we observe that these profits fall as $\gamma$ is rising, that is, gains from collusion are higher. As collusive prices are set uniformly and hence optimal deviation only depends on a firm’s private signal, collusive and deviating profits remain unchanged compared to the symmetric-signal case. We therefore arrive at the following proposition:

**Proposition 4.** For any $\gamma > 0$, the critical discount factor $\bar{\delta}_{\text{cor}}$ is strictly lower compared to the case of independent signal quality $\sigma$. In addition, $\bar{\delta}_{\text{cor}}$ is non-monotone in $\sigma$.

*Proof.* See Appendix A.

At the lower and upper bound of $\sigma$, the cases of correlated and independent signals are equivalent by construction and hence the critical discount factors are equal. The following corollary directly results from Propositions 2 and 4:

**Corollary 2.** For any $\gamma > 0$, the probability that a ban on price discrimination facilitates collusion is strictly lower compared to the case of independent signal quality $\sigma$. Furthermore, the difference strictly increases in $\gamma$.

5. Conclusion

The use of big data—especially consumer data—for pricing strategies have substantially increased in recent times. Consumer data is only an estimation of a consumer’s preferences. Hence, imprecision is an important factor when firms make their pricing decisions.

In this paper, we focus on these developments in particular in online markets. Specifically, we look at the impact of data-driven price-discrimination strategies on firms’ incentives to collude. In our setup, two competing firms obtain imperfect private signals on consumer preferences. We find that an increase in signal precision results in a non-monotonic change in firms’ ability to sustain collusion. In particular, for low levels of signal precision, collusion is facilitated with an increase in signal precision. On the other hand, for sufficiently high precision levels, an increase in signal precision makes collusion harder to sustain. We also find that the critical discount factor is non-monotone in transport costs.

Compared to a scenario in which price discrimination is banned, we find that for low levels of signal precision, collusion is easier to sustain under price discrimination than under no price discrimination. For sufficiently high levels of signal precision, we find that collusion is harder to sustain under a discriminatory pricing regime than under a ban on price discrimination.
With regard to policy implications, the results from our study give rise to the following conclusions. When thinking about banning price discrimination and its effects on firms’ possibility to collude, one must keep in mind that the outcome crucially depends on signal precision/data quality (and differentiation). Under competition an increase in signal precision leads to lower consumer prices due to best-response asymmetries. Also, when signals are perfect, the likelihood of collusive behavior is lowest. Both effects are good news for consumers. However, marginal increases in signal precision can have the undesired effect of stabilizing collusion and hence higher prices. At the same time, the model we employ does not allow to draw conclusions with regard to welfare, as we do not take into account consumer preferences for privacy or other adverse effects due to discrimination of consumers.

We can also conclude that the positive view in the competition-policy debate with regard to third-degree price discrimination and collusion appears to depend on whether information is perfect or not: Whereas allowing for price discrimination under perfect information unambiguously helps to make (tacit) collusion harder to sustain, the opposite can be true under imperfect private signals. On a more general note and related to the above-mentioned aspect, one may argue that when the exchange of consumer data leads to higher signal precision towards perfect information, competition authorities should be less concerned with regard to collusive activity than in the case in which firms exchange data on prices, demands, etc.

References


Appendix A

Proof of Lemma 2. Without loss of generality, suppose that firm B sets the collusive price and firm A deviates unilaterally. As firm B charges $p^c$ regardless of its signal, we have both $\tilde{x}_1 = \tilde{x}_3$ and $\tilde{x}_2 = \tilde{x}_4$. Substituting this into Equations (3) and (4), firm A expects its demand conditional on receiving signal $s_L$ to be

$$D_{A,L} = \sigma + 2(1-\sigma)\left(\tilde{x}_1 - \frac{1}{2}\right), \quad (6)$$

and its demand conditional on receiving signal $s_R$ to be

$$D_{A,R} = (1-\sigma) + 2\sigma\left(\tilde{x}_4 - \frac{1}{2}\right). \quad (7)$$

Then, the profit maximization problem of firm A is given as

$$\max_{p^d_{A,L},p^d_{A,R}} \mathbb{E} \left[ \pi^d_A \right] = \frac{1}{2} \left( p^d_{A,L} D_{A,L} + p^d_{A,R} D_{A,R} \right),$$

with $p_{B,L} = p_{B,R} = p^c$. Taking first order conditions with respect to firm A’s deviation prices, we get inner solutions

$$p^*_d_{A,L} = \frac{1}{2} + \frac{\tau(3\sigma - 1)}{4(1 - \sigma)} \quad \text{and} \quad p^*_d_{A,R} = \frac{1}{2} - \frac{3\tau(3\sigma - 2)}{4\sigma}.$$

Using these, we make the following observations:

- $\tau > \frac{2(1-\sigma)}{5-3\sigma} =: \tau_1 \implies \tilde{x}_1 < 1,$
• $\tau > \frac{2\sigma}{2 + 3\sigma} =: \tau_2 \implies \tilde{x}_2 < 1$,

• $\tau < \frac{2(1 - \sigma)}{1 + \sigma} =: \tau_3 \implies p^d_{A,L} < p^c$,

where $p^c = 1 - \tau/2$. Thereby, it holds that $\tau_3 > \tau_2$ if and only if $\sigma < 1/\sqrt{2}$. Consequently, for $\sigma < 1/\sqrt{2}$, we obtain the order of parameters $0 < \tau_1 < \tau_2 < \tau_3 < 2/3$. On the other hand, for $\sigma > 1/\sqrt{2}$, we obtain the order of parameters $\tau_1 < \tau_3 < \tau_2 < 2/3$. In the following, we determine the optimal deviation behavior of firm $A$ conditional on $\tau$ by distinguishing the following five cases:

Case (i): For $\tau \leq \tau_1$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 = \tilde{x}_2 = 1$ in order to take over the whole market, that is, $p^d_{A,L} = p^d_{A,R} = 1 - 3\tau/2$. Thereby, its conditional expected demand as defined in Equations (6) and (7) is equal to $1/2$ regardless of the signal. Then, the expected profit from deviating is given by

$$\pi^d_A = 1 - \frac{3\tau}{2}.$$ 

Case (ii): For $\tau_1 < \tau \leq \min\{\tau_2, \tau_3\}$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 < \tilde{x}_2 = 1$, that is, $p^d_{A,L} = p^d_{A,R} = 1 - 3\tau/2$. Substituting this into Equations (6) and (7), we get an expected deviation profit of

$$\pi^d_A = \frac{(3\tau(1 + \sigma) + 2(1 - \sigma))^2 - 32\tau^2}{32\tau(1 - \sigma)}.$$ 

Case (iii): Suppose $\sigma \leq 1/\sqrt{2}$. For $\tau_2 < \tau \leq \tau_3$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 < \tilde{x}_2 < 1$, that is, $p^d_{A,L} = p^d_{A,R} = 1 - 3\tau/2$. Substituting this into Equations (6) and (7), we get an expected deviation profit of

$$\pi^d_A = \frac{\tau}{8\sigma(1 - \sigma)} + \frac{4(\tau + 1) - 15\tau^2}{32\tau}.$$ 

Case (iv): For now suppose $\sigma > 1/\sqrt{2}$. For $\tau_3 < \tau \leq \tau_2$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 < \tilde{x}_2 = 1$, that is, $p^d_{A,L} = p^c$ and $p^d_{A,R} = 1 - 3\tau/2$. By Assumption 1, firm $A$ does not find it profitable to charge more than $p^c$ from its loyal consumers as long as firm $B$ uniformly charges $p^c$. Substituting this into Equations (6) and (7), we get
an expected deviation profit of

$$\pi^d_A = \frac{2 - 3\tau + \sigma(2 - \tau)}{4}. $$

Case (v): For $\tau > \max\{\tau_2, \tau_3\}$, we infer from our observations above that firm $A$ optimally sets prices such that $\bar{x}_1, \bar{x}_2 < 1$, that is, $p_{A,L} = p^c$ and $p_{A,R}^d = p_{A,L}^d$. For the same reason as before, by Assumption 1, firm $A$ charges $p^c$ from its loyal consumers as long as firm $B$ uniformly charges $p^c$. Substituting this into Equations (6) and (7), we get an expected deviation profit of

$$\pi^d_A = \frac{\tau^2(2 - \sigma)^2 + 4\sigma^2(\tau + 1) + 8\sigma(1 - \tau)}{32\tau\sigma}. $$

Now, it is straightforward to check that for both prices and deviation profits their respective left-hand and right-hand limits for $\sigma$ and $\tau$ approaching the bounds of Case (i)--(iv) from above are equal. Hence, they are continuous.

Further, it is straightforward to check that there are no kinks in both prices and deviation profits since the respective left-hand and right-hand limits of their derivatives for $\sigma$ and $\tau$ approaching the bounds of Case (i)--(iv) are equal. Hence, they are differentiable. \hfill \Box

Proof of Proposition 1. Taking the collusive profits from Lemma 1, the deviation profits from Lemma 2 and the punishment profits as given in Section 3.1, we can solve for the critical discount factor as defined in Condition (2). As only the functional form of the deviation payoff is changing with $\tau$, we distinguish the five cases as defined in the proof of Lemma 2, that is:

Case (i): For $\tau \leq \tau_1$, we get

$$\bar{\delta} = \frac{\sigma(2\sigma + 1)^2(2 - 5\tau)}{(2\sigma + 1)^2(4\sigma - 6\sigma\tau) - 2(4\sigma^2 + 1)\tau}. $$

Case (ii): For $\tau_1 < \tau \leq \min\{\tau_2, \tau_3\}$, we get

$$\bar{\delta} = \frac{2\sigma(2\sigma + 1)^2((\sigma(3\tau - 2) + 3\tau + 2)^2 + (8\tau - 16)(\tau - \sigma\tau) - 32\tau^2)}{2\sigma(2\sigma + 1)^2((\sigma(3\tau - 2) + 3\tau + 2)^2 - 32\tau^2) - 32(4\sigma^2 + 1)\tau(\tau - \sigma\tau)}. $$

Case (iii): Suppose $\sigma \leq 1/\sqrt{2}$. For $\tau_2 < \tau \leq \tau_3$, we get

$$\bar{\delta} = \frac{((\sigma(3\tau - 2) + 3\tau + 2)^2 + (8\tau - 16)(\tau - \sigma\tau) - 32\tau^2)}{((\sigma(3\tau - 2) + 3\tau + 2)^2 - 32\tau^2) - 32(4\sigma^2 + 1)\tau(\tau - \sigma\tau)}. $$
Case (iv): Now suppose $\sigma > 1/\sqrt{2}$. For $\tau_3 < \tau \leq \tau_2$, we get
\[
\bar{\delta} = \frac{\sigma(2\sigma + 1)^2(\sigma(\tau - 2) + 2\tau)}{(\sigma(\sigma(4\sigma + 4) + 21) + 3) + 2)\tau - 2\sigma(\sigma + 1)(2\sigma + 1)^2}.
\]

Case (v): For $\tau > \max\{\tau_2, \tau_3\}$, we get
\[
\bar{\delta} = \frac{(2\sigma + 1)^2(\sigma(\tau + 2) - 2\tau)^2}{C},
\]
where $C := 4\tau(\sigma^2(2\sigma + 3)^2 + \sigma\tau) + (\sigma^2(4(\sigma - 11)\sigma - 95) - 12)\tau^2 + 4\sigma^2(2\sigma + 1)^2 + 8\sigma\tau$. From continuity and differentiability of all profit functions entering Condition (2)—namely $\pi^*, \pi^c, \pi^d$—continuity and differentiability of $\bar{\delta}$ with respect to $\sigma$ and $\tau$ immediately follows.

In order to do comparative statics, we take the derivative of $\bar{\delta}$ as defined in Condition (2) with respect to $\sigma$, that is
\[
\frac{\partial \bar{\delta}}{\partial \sigma} = \frac{\partial \pi^d}{\partial \sigma}(\pi^c - \pi^*) + \frac{\partial \pi^*}{\partial \sigma}(\pi^d - \pi^c)
\]

We observe that
\[
\frac{\partial \bar{\delta}}{\partial \sigma} \geq 0 \iff \frac{\partial \pi^d}{\partial \sigma}(\pi^c - \pi^*) + \frac{\partial \pi^*}{\partial \sigma}(\pi^d - \pi^c) \geq 0.
\]

Exploiting this, we show that $\partial \bar{\delta} / \partial \sigma |_{\sigma = 1/2} < 0$ and $\partial \bar{\delta} / \partial \sigma |_{\sigma = 1} > 0$ in all relevant cases. Figure 1 nicely illustrates which parameter ranges of $\tau$ have to be considered for the respective extreme value of $\sigma$. For $\sigma = 1/2$, we have $\tau_1 = \tau_2 = 2/7 < \tau_3 = 2/3$. For $\sigma = 1$, we have $\tau_1 = \tau_3 = 0 < \tau_2 = 2/5$. We obtain the following:

- If $\sigma = 1/2$, we observe that
  - for $\tau \in \left(0, \frac{2}{7}\right]$, $\frac{\partial \bar{\delta}}{\partial \sigma} |_{\sigma = 1/2} = \frac{4\tau(5\tau - 2)}{(4 - 8\tau)^2} < 0$,
  - for $\tau \in \left(\frac{2}{7}, \frac{2}{3}\right]$, $\frac{\partial \bar{\delta}}{\partial \sigma} |_{\sigma = 1/2} = -\frac{32\tau^2}{(5\tau + 2)^2} < 0$.

- If $\sigma = 1$, we observe that
  - for $\tau \in \left(0, \frac{2}{5}\right]$, $\frac{\partial \bar{\delta}}{\partial \sigma} |_{\sigma = 1} = \frac{9(3\tau - 2)}{40\tau - 36} > 0$,
  - for $\tau \in \left(\frac{2}{5}, \frac{2}{3}\right]$, $\frac{\partial \bar{\delta}}{\partial \sigma} |_{\sigma = 1} = \frac{24(\tau - 2)\tau(149\tau - 4) - 108}{(\tau(149\tau - 108) - 36)^2} > 0$.

Hence, $\bar{\delta}$ is non-monotonic with respect to $\sigma$. 

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In order to do comparative statics of $\delta$ with respect to $\tau$, we apply the implicit function theorem to the binding case of Inequality (1). We get

$$\frac{\partial \delta}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \pi^d - \pi^e \right) + \frac{\partial}{\partial \tau} \left( \pi^* - \pi^d \right) \delta.$$  

Exploiting that $\pi^d > \pi^*$, the sign of the above expression only depends on the sign of the numerator. It is straightforward to verify that the numerator is strictly negative in Case (i)-(iv) as defined in the proof of Lemma 2. Only in Case (v) the sign of the numerator can change. Solving for $\tau$, we get

$$\left( \frac{\partial \pi^d}{\partial \tau} - \frac{\partial \pi^e}{\partial \tau} \right) \left( 1 - \delta \right) + \delta \frac{\partial (\pi^* - \pi^e)}{\partial \tau} < 0 \iff \tau < \frac{2\sigma(2\sigma + 1)^2}{\sigma(4\sigma(3\sigma + 5) - 5) + 2} =: \tilde{\tau}.$$  

We observe that $\tilde{\tau} \in (\max\{\tau_2, \tau_3\}, 2/3)$. Given this, we conclude that for $\tau \in (\max\{\tau_2, \tau_3\}, \tilde{\tau})$, the numerator is negative and hence it holds true that $\partial \delta / \partial \tau < 0$. For $\tau \in (\tilde{\tau}, 2/3]$, the numerator is positive and hence it holds true that $\partial \delta / \partial \tau > 0$. Finally, the numerator is zero at $\tau = \tilde{\tau}$ and hence it holds true that $\partial \delta / \partial \tau = 0$. \qed

**Proof of Proposition 2.** We define $\tilde{\sigma}(\tau)$ as the signal precision levels that solve

$$\tilde{\delta}_{\text{ban}} = \tilde{\delta}$$  

(8) for a given $\tau$. Figure 2 displays all parameter combinations $(\sigma, \tau)$ such that Equation (8) is satisfied. By construction, $\sigma = 1/2$ solves Equation (8) for any $\tau$ because the signal is uninformative and firms cannot price-discriminate, that is, $\tilde{\sigma}(\tau) = 1/2$ for $\tau [0, 2/3]$. If $\tau = 0$, there is perfect competition and price discrimination is not feasible, that is, $\tilde{\sigma}(0) = \{\sigma \in [1/2, 1]\}$. Further, we have $\tilde{\sigma}(2/3) = 1/2$. If the level of transport costs is maximal and the signal uninformative, firms cannot benefit from collusion, hence the critical discount factor is zero by definition. However, if $\sigma$ is increasing, temptation to deviate always outweighs gains from collusion, that is, $\tilde{\delta} > \tilde{\delta}_{\text{ban}}$. For any $\tau \in (0, 2/3)$, we can see from Figure 2 that there always exists a $\tilde{\sigma}(\tau) \in (1/2, 1)$. For any such $\tilde{\sigma}(\tau)$, it holds that $\tilde{\delta} \leq \tilde{\delta}_{\text{ban}}$ if and only if $\sigma \leq \tilde{\sigma}(\tau)$. \qed

**Proof of Proposition 3.** As payoffs under collusion remain unchanged, we are left with determining punishment and deviation payoffs. Then, we compute the critical discount factor $\tilde{\delta}_{\text{asy}}$. Finally, we show that the critical discount factor is always increasing in signal asymmetry compared to the symmetric case.

Let's first determine punishment payoffs. Given beliefs as derived in Section 4.1, we obtain expected demand of firm A conditional on receiving signals $s_L$.
and \( s_R \), respectively, of
\[
D_A(p_{A,L}, p_{B,L}, p_{B,R}|s_L) = \frac{1}{\sigma_1 + 1 - \sigma_2} \times \left( 2\sigma_1\sigma_2 \tilde{x}_1 + \sigma_1(1 - \sigma_2) + 2(1 - \sigma_1)\sigma_2 \left( \tilde{x}_2 - \frac{1}{2} \right) \right)
\]
(9)
and
\[
D_A(p_{A,L}, p_{B,L}, p_{B,R}|s_R) = \frac{1}{\sigma_2 + 1 - \sigma_1} \times \left( 2(1 - \sigma_1)\sigma_2 \tilde{x}_3 + (1 - \sigma_1)(1 - \sigma_2) + 2\sigma_2\sigma_1 \left( \tilde{x}_4 - \frac{1}{2} \right) \right),
\]
(10)
with \( \tilde{x}_1 - \tilde{x}_4 \) referring to the indifferent consumers as defined in the main analysis. Firm A’s maximization problem is then defined as in Equation (5). Firm B’s maximization problem is determined analogously. Solving first-order conditions with respect to prices simultaneously, we get optimal prices
\[
p_{A,L}^* = p_{B,R}^* = \frac{2\tau\sigma_1}{\sigma_2 + 2\sigma_1\sigma_2} \quad \text{and} \quad p_{A,R}^* = p_{B,L}^* = \frac{\tau}{\sigma_2 + 2\sigma_1\sigma_2},
\]
where \( p_{A,R}^* < p_{A,L}^* \) and \( p_{B,L}^* < p_{B,R}^* \) as long as the signal is informative. The resulting equilibrium profit for each firm amounts to
\[
\pi^* = \frac{\tau(1 + 4\sigma_1^2)}{2\sigma_2(1 + 2\sigma_1)^2}.
\]
Next, let’s determine deviation payoffs. Without loss of generality, suppose that firm B sets the collusive price and firm A deviates unilaterally. As firm B charges \( p^c \) regardless of its signal, we have both \( \tilde{x}_1 = \tilde{x}_3 \) and \( \tilde{x}_2 = \tilde{x}_4 \). Substituting this into Equations (9) and (10), firm A expects its demand conditional on receiving signal \( s_L \) to be
\[
D_{A,L} = \frac{1}{\sigma_1 + 1 - \sigma_2} \left( \sigma_1 + 2(1 - \sigma_1) \left( \tilde{x}_1 - \frac{1}{2} \right) \right),
\]
(11)
and its demand conditional on receiving signal \( s_R \) to be
\[
D_{A,R} = \frac{1}{\sigma_2 + 1 - \sigma_1} \left( (1 - \sigma_2) + 2\sigma_2 \left( \tilde{x}_2 - \frac{1}{2} \right) \right).
\]
(12)
Then, the profit maximization problem of firm $A$ is given as

$$
\max_{p_{A,L}, p_{A,R}} \mathbb{E}[\pi^d_A] = \frac{1}{2} \left( \frac{p_{A,L}^d D_{A,L}}{\sigma_1 + 1 - \sigma_2} + \frac{p_{A,R}^d D_{A,R}}{\sigma_2 + 1 - \sigma_1} \right),
$$

with $p_{B,L} = p_{B,R} = p^c$. Taking first order conditions with respect to firm $A$’s deviation prices, we get inner solutions

$$
p_{A,L}^d = \frac{1}{2} + \frac{\tau (3 \sigma_1 - 1)}{4(1 - \sigma_1)} \quad \text{and} \quad p_{A,R}^d = \frac{1}{2} - \frac{\tau (3 \sigma_2 - 2)}{4 \sigma_2}.
$$

Using these, we make the following observations:

- $\tau > \frac{2(1 - \sigma_1)}{5 - 3\sigma_1} =: \tau_1 \implies \tilde{x}_1 < 1$,
- $\tau > \frac{2 \sigma_2}{2 + 3 \sigma_2} =: \tau_2 \implies \tilde{x}_2 < 1$,
- $\tau < \frac{2(1 - \sigma_1)}{1 + \sigma_1} =: \tau_3 \implies p_{A,L}^d < p^c$,

where $p^c = 1 - \tau/2$. The thresholds are ordered as $\tau_1 < \tau_2 < \tau_3$ if $\sigma_1 < (1 + \sigma_2)/(1 + 2 \sigma_2)$ and $\sigma_2 < 1/\sqrt{2}$. Else, thresholds are ordered as $\tau_1 < \tau_3 < \tau_2$. In the following, we determine the optimal deviation behavior of firm $A$ conditional on $\tau$ by distinguishing the following five cases:

Case (i): For $\tau \leq \tau_1$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 = \tilde{x}_2 = 1$ in order to take over the whole market, that is, $p_{A,L}^d = p_{A,R}^d = 1 - 3\tau/2$. Thereby, its conditional expected demand as defined in Equations (11) and (12) is equal to $1/2$ regardless of the signal. Then, the expected profit from deviating is given by

$$
\pi^d_A = 1 - \frac{3\tau}{2}.
$$

Case (ii): For $\tau_1 < \tau \leq \min\{\tau_2, \tau_3\}$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 < \tilde{x}_2 = 1$, that is, $p_{A,L}^d = p_{A,R}^d = 1 - 3\tau/2$. Substituting this into Equations (11) and (12), we get an expected deviation profit of

$$
\pi^d_A = \frac{(3\tau(1 + \sigma_1) + 2(1 - \sigma_1))^2 - 32\tau^2}{32\tau(1 - \sigma_1)}.
$$

Case (iii): Suppose $\sigma_1 < (1 + \sigma_2)/(1 + 2 \sigma_2)$ and $\sigma_2 \leq 1/\sqrt{2}$. For $\tau_2 < \tau \leq \tau_3$, we infer from our observations above that firm $A$ optimally sets prices such
that $\tilde{x}_1, \tilde{x}_2 < 1$, that is, $p^d_{A,L} = p^c_{A,L}$ and $p^d_{A,R} = p^c_{A,L}$. Substituting this into Equations (11) and (12), we get an expected deviation profit of

$$\pi_A^d = \frac{D}{32(\sigma_1 - 1)\sigma_2 \tau},$$

where $D := 4(\sigma_1 - 1)(\tau(3\sigma_1 - 3\sigma_2 + 1)\sigma_2(-\sigma_1 + \sigma_2 + 1)) + \tau^2(9(\sigma_1 - 1)\sigma_2^2 - 3\sigma_1(3\sigma_1 + 2)\sigma_2 + 4\sigma_1 + 11\sigma_2 - 4)\sigma_2$.

Case (iv): Suppose $\sigma_1 \geq (1 + \sigma_2)/(1 + 2\sigma_2)$ and $\sigma_2 > 1/\sqrt{2}$. For $\tau_3 < \tau \leq \tau_2$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1 < \tilde{x}_2 = 1$, that is, $p_{A,L} = p^c$ and $p^d_{A,R} = \frac{1 - 3\tau}{2}$. By Assumption 1, firm $A$ does not find it profitable to charge more than $p^c$ from its loyal consumers as long as firm $B$ uniformly charges $p^c$. Substituting this into Equations (11) and (12), we get an expected deviation profit of

$$\pi_A^d = \frac{2 - 3\tau + \sigma_1(2 - \tau)}{4}.$$

Case (v): For $\tau > \max\{\tau_2, \tau_3\}$, we infer from our observations above that firm $A$ optimally sets prices such that $\tilde{x}_1, \tilde{x}_2 < 1$, that is, $p_{A,L} = p^c$ and $p^d_{A,R} = p^c_{A,L}$. For the same reason as before, by Assumption 1, firm $A$ charges $p^c$ from its loyal consumers as long as firm $B$ uniformly charges $p^c$. Substituting this into Equations (11) and (12), we get an expected deviation profit of

$$\pi_A^d = \frac{1}{32}(16\sigma_1 - 4(2\sigma_1 + 3)\tau + \frac{\sigma_2(2 - 3\tau)^2}{\tau} + \frac{4\tau}{\sigma_2} + 8).$$

Now, it is straightforward to check that for both prices and deviation profits their respective left-hand and right-hand limits for $\sigma_1, \sigma_2$ and $\tau$ approaching the bounds of Case (i)–(iv) from above are equal. Hence, they are continuous.

Further, it is straightforward to check that there are no kinks in both prices and deviation profits since the respective left-hand and right-hand limits of their derivatives for $\sigma_1, \sigma_2$ and $\tau$ approaching the bounds of Case (i)–(iv) are equal. Hence, they are differentiable.

Taking the collusive profits from Lemma 1, the deviation profits from the above analysis and the punishment profits as given in Section 4.1, we can solve for the critical discount factor as defined in Equation (2). As only the functional form of the deviation payoff is changing with $\tau$, we distinguish the five cases as defined in the proof of Lemma 2, that is:

Case (i): For $\tau \leq \tau_1$, we get

$$\delta_{asy} = \frac{(2\sigma_1 + 1)^2\sigma_2(5\tau - 2)}{2(4\sigma_1^2 + 1)\tau + 2(2\sigma_1 + 1)^2\sigma_2(3\tau - 2)}.$$
Case (ii): For \( \tau_1 < \tau \leq \min\{\tau_2, \tau_3\} \), we get

\[
\tilde{\delta}_{asy} = \frac{(2\sigma_1 + 1)^2 \sigma_2 (-15 \tau^2 + \sigma_1^2 (2 - 3 \tau)^2 + 2 \sigma_1 (\tau + 2)(5 \tau - 2) - 4 \tau + 4)}{E},
\]

where \( E := (2\sigma_1 + 1)^2 \sigma_2 (4(\sigma_1 - 1)^2 + (9\sigma_1 (\sigma_1 + 2) - 23) \tau^2 - 12(\sigma_1^2 - 1) \tau) + 16(\sigma_1 - 1) (4\sigma_1^2 + 1) \tau^2). \)

Case (iii): Suppose \( \sigma_1 < (1 + \sigma_2)/(1 + 2\sigma_2) \) and \( \sigma_2 \leq 1/\sqrt{2} \). For \( \tau_2 < \tau \leq \tau_3 \), we get

\[
\tilde{\delta}_{asy} = \frac{F}{G}
\]

where \( F := (2\sigma_1 + 1)^2 (\sigma_2 (4(\sigma_1 - 1)^2 + (9\sigma_1 (\sigma_1 - 2) - 3) \tau^2 - 12(\sigma_1 - 1)^2 \tau) - 4(\sigma_1 - 1)^2 \tau^2 - (\sigma_1 - 1) \sigma_2^2 (2 - 3 \tau)^2), \) and \( G := (2\sigma_1 + 1)^2 \sigma_2 (-11 \tau^2 + \sigma_1^2 (2 - 3 \tau)^2 + 2 \sigma_1 (\tau + 2)(3 \tau - 2) + 4 \tau + 4 - (\sigma_1 - 1) \sigma_2 (2 - 3 \tau)^2) + 4(\sigma_1 - 1) (4\sigma_1 (3\sigma_1 - 1) + 3) \tau^2). \)

Case (iv): Suppose \( \sigma_1 \geq (1 + \sigma_2)/(1 + 2\sigma_2) \) and \( \sigma_2 > 1/\sqrt{2} \). For \( \tau_3 < \tau \leq \tau_2 \), we get

\[
\tilde{\delta}_{asy} = \frac{(2\sigma_1 + 1)^2 \sigma_2 ((\sigma_1 + 2) \tau - 2 \sigma_1)}{2 (4\sigma_1^2 + 1) \tau + (2\sigma_1 + 1)^2 \sigma_2 ((\sigma_1 + 3) \tau - 2(\sigma_1 + 1))}.
\]

Case (v): For \( \tau > \max\{\tau_2, \tau_3\} \), we get

\[
\tilde{\delta}_{asy} = \frac{(2\sigma_1 + 1)^2 (4 \tau^2 - 4 \sigma_2 \tau (2 \sigma_1 (\tau - 2) + \tau + 2) + \sigma_2^2 (2 - 3 \tau)^2)}{H},
\]

where \( H := (2\sigma_1 + 1)^2 (\sigma_2^2 (2 - 3 \tau)^2 + \sigma_2 \tau (16 \sigma_1 - 4(\sigma_1 + 3) \tau + 8)) - 16 \sigma_1 (3\sigma_1 - 1) + 3) \tau^2). \)

From continuity and differentiability of all profit functions entering Condition (2)—namely \( \pi^*, \pi^c, \pi^d \)—continuity of \( \tilde{\delta} \) with respect to \( \sigma_1, \sigma_2 \) and \( \tau \) immediately follows.

Using this, we show that \( \tilde{\delta}_{asy} \geq \max\{\tilde{\delta}(\sigma = \sigma_1), \tilde{\delta}(\sigma = \sigma_2)\} \). For this to hold, it is sufficient that \( \partial \tilde{\delta}_{asy}/\partial \sigma^1|_{\sigma_1 = \sigma_2 = \sigma} > 0 \) and \( \partial \tilde{\delta}_{asy}/\partial \sigma^2|_{\sigma_1 = \sigma_2 = \sigma} < 0 \). Why is this? Starting from the symmetric case, asymmetry can be created by either \( \sigma_1 > \sigma \) or \( \sigma_2 < \sigma \). Straightforward calculations immediately verify that the stepwise derivatives of \( \tilde{\delta}_{asy} \) actually satisfy the sufficient conditions.

Finally, we show that \( \tilde{\delta}_{asy} \) is non-monotonic in \( \sigma_1 \) and \( \sigma_2 \). At \( \sigma_1 = \sigma_2 = \frac{1}{2} \) and \( \sigma_1 = \sigma_2 = 1 \), we have \( \tilde{\delta}_{asy} = \tilde{\delta} \) by construction. Hence, by exploiting Proposition 2, it is sufficient to show that \( \tilde{\delta}_{asy} \) is decreasing around the lower bound of its support. By evaluating the relevant cases, we obtain the following:

- for \( \tau \in \left[0, \frac{2}{7}\right] \), \( \frac{\partial \tilde{\delta}}{\partial \sigma_1}|_{\sigma_1 = \sigma_2 = \frac{1}{2}} + \frac{\partial \tilde{\delta}}{\partial \sigma_2}|_{\sigma_1 = \sigma_2 = \frac{1}{2}} = \frac{\tau (5 \tau - 2)}{4(1 - 2 \tau)^2} < 0 \),

- for \( \tau \in \left(\frac{2}{7}, \frac{2}{3}\right] \), \( \frac{\partial \tilde{\delta}}{\partial \sigma_1}|_{\sigma_1 = \sigma_2 = \frac{1}{2}} + \frac{\partial \tilde{\delta}}{\partial \sigma_2}|_{\sigma_1 = \sigma_2 = \frac{1}{2}} = - \frac{32 \tau^2}{(5 \tau + 2)^2} < 0 \),

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where \( \tau_1 = \tau_2 = 2/7 \) and \( \tau_3 = 2/3 \) for \( \sigma = 1/2 \). By continuity of \( \bar{\delta}_{\text{asy}} \), there exist \( \sigma_1 > \sigma_2 \geq 1/2 \), such that the above signs of the derivative continue to hold.

Proof of Proposition 4. As collusion and deviation payoffs remain unchanged, we are left with determining punishment payoffs. Then, we argue how \( \bar{\delta}_{\text{asy}} \) is affected.

Let's determine punishment payoffs. Given beliefs as derived in Section 4.2, we obtain expected demand of firm A conditional on receiving signals \( s_L \) and \( s_R \), respectively, of

\[
D_A(p_{A,L}, p_{B,L}, p_{B,R} | s_L) = 2(\sigma^2 + \gamma)\tilde{x}_1 \\
+ (\sigma(1 - \sigma) - \gamma) + 2((1 - \sigma)\sigma - \gamma) \left( \tilde{x}_2 - \frac{1}{2} \right)
\]

and

\[
D_A(p_{A,L}, p_{B,L}, p_{B,R} | s_R) = 2(\sigma^2 + \gamma) \left( \tilde{x}_4 - \frac{1}{2} \right) \\
+ 2((1 - \sigma)\sigma - \gamma) \tilde{x}_3 + ((1 - \sigma)^2 + \gamma),
\]

with \( \tilde{x}_1 - \tilde{x}_4 \) referring to the indifferent consumers from above. Expected profits of firm A are then defined as in (5), and the decision problem of firm B is derived analogously. Solving first-order conditions with respect to prices simultaneously, we get optimal prices

\[
\frac{\partial \pi^*}{\partial \gamma} = -\frac{(1 - 2\sigma)^2 \sigma \tau}{2(2\gamma + 2\sigma^2 + \sigma)^3} < 0 \quad \forall \gamma \in [0, \sigma(1 - \sigma)], \sigma \in \left(0, \frac{1}{2}\right), \tau \in \left(0, \frac{2}{3}\right).
\]

We further observe, that collusion and deviation payoffs only depend on a firm’s private signal and hence are defined as in Section 3. It follows immediately from the definition of the critical discount factor in 2 that the lower is the punishment payoff, the less patient players have to be in order to sustain collusion. Hence, we conclude that for any \( \sigma \) and \( \gamma > 0 \), we have \( \bar{\delta}_{\text{cor}} < \bar{\delta} \). In addition, \( \bar{\delta}_{\text{cor}} \) is continuous in \( \sigma \).

Finally, we show that \( \bar{\delta}_{\text{cor}} \) is non-monotonic in \( \sigma \). At \( \sigma = 1/2 \) and \( \sigma = 1 \), we have \( \bar{\delta}_{\text{cor}} = \bar{\delta} \) by construction. Hence, from the above observations and Proposition 2, the non-monotonicity immediately follows. \( \square \)
Appendix B

In this section, we characterize an optimal penal code. The game and strategy profile is as described above except for punishment. In order to derive optimal penal codes, we first need to determine the minmax payoff of firm \( i = 1, 2 \)—the stick. Due to positive transport costs and strategic complementarity, the worst firm \( j \neq i \) can do to \( i \) is charging \( p^o := 0 \) irrespectively of its private signal \( s_j \). Given this, we can specify beliefs over consumers’ preferences and the relevant indifferent consumers analogously to Section 3. Firm \( i \) faces the following optimization problem:

\[
\max_{p_{i,L}, p_{i,R}} E[\pi_i] = \frac{1}{2} \left( \sigma p_{i,L} \left( 1 - \frac{P_{A,L}}{\tau} \right) + (1 - \sigma) p_{i,R} \left( 1 - \frac{P_{A,R}}{\tau} \right) \right).
\]

As the objective function is concave, the optimal solution is \( p_{i,L} = p_{i,R} = \tau/2 =: p^{mx} \). Then, firm \( i \)'s minmax payoff is given by

\[
\pi^{mx} = \frac{\tau}{8}.
\]

Next, we have to make sure that it is incentive compatible for firm \( j \) to punish firm \( i \) after observing a deviation from charging the collusive price—the carrot. As punishment is costly for firm \( j \), it has to be compensated after charging a zero price for \( T \) periods. In our game, the most efficient compensation is reversion to collusive behavior as defined in Lemma 1, which provides each firm with payoff \( \pi^c \). First, we need to find the minimum amount of punishment periods \( T^* \) such that punishment is incentive compatible for any discount factor \( \delta \). Observing that punishment is most efficient if the deviator charges \( p^o \) as well throughout the respective \( T \) periods, we define the following punishment strategy profile:

- If firm \( j \) observes an unexpected deviation of firm \( i \) from \( p^c \) in any period \( t \), both firms charge \( p^o \) in periods \( t + 1 \) to \( t + T^* \). Then,
  - if a firm deviates from \( p^o \) in any period \( t' \in \{t+1, \ldots, t+T^*\} \), both firms charge \( p^o \) in periods \( t' + 1 \) to \( t' + T^* \), and
  - if there is no deviation from \( p^o \) throughout \( T^* \) periods, both firms charge \( p^c \) again.

To see why this is optimal, let’s define \( T^* \) such that a firm is indifferent between the following scenarios: (i) receiving zero payoffs for \( T \) periods and afterwards receiving \( \pi^c \) for the rest of the game; and (ii) deviating to \( p^{mx} \) in period \( t \),

\( ^{20} \)One can easily verify, that the critical discount factor is strictly larger when allowing the deviator to receive minmax payoffs during punishment phase.
receiving zero payoffs for $T$ periods and afterwards receiving $\pi^c$ for the rest of the game. Hence, $T^*$ solves

$$V^p = \pi^{mx} + \delta V^p,$$

where $V^p := 0 + \delta 0 + \ldots + \delta^{T-1} 1 + \delta^T \pi^c$. The implicit solution is given by

$$\delta T^* = \frac{\pi^{mx}}{\pi^c}.$$  

As $\delta \in (0, 1)$, $\pi^{mx}$ is bounded from above, and $\pi^c$ is bounded away from zero, $T^*$ is finite for any $\tau > 0$. We observe that the larger $\delta$, the larger $T^*$. The intuition behind this trade-off is that the more patient firms are, the more tempted they are to trade $\pi^{mx}$ in period $t$ against delaying the future stream of $\pi^c$ by a single period.

Finally, we substitute for the payoff stream from optimal penal codes $V^p$ in Inequality (1) to get the following condition for OSDP to hold:

$$\frac{\pi^c}{1 - \delta} \geq \pi^d + \frac{\delta \left( \delta T^* \pi^c \right)}{1 - \delta}.$$  

Substituting for the implicit characterization of $T^*$, we obtain

$$\delta \geq \frac{\pi^d - \pi^c}{\pi^d - \pi^{mx}} =: \tilde{\delta}_{mx}.$$  

It is easily verified that $\tilde{\delta}_{mx} < \tilde{\delta}$ as $\pi^{mx} < \pi^*$ for all $\sigma$ and $\tau$.\footnote{Moreover, $\tilde{\delta}_{mx}$ is lower compared to the critical discount factors in case of asymmetric signal quality and correlated signals.} Since $V^p$ is independent of signal quality, $\tilde{\delta}_{mx}$ always rises in $\sigma$.  

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